PHILOSOPHICAL TRANSACTIONS.

IX. THE BAKERIAN LECTURE.—*On the Theory of the Astronomical Refractions.* By

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THE apparent displacement of the stars caused by the inflection of light in its passage through the atmosphere, is treated by the astronomer like most other irregularities which he has occasion to consider. A set of mean quantities is first provided; and the occasional deviations of the true places from the mean are ascertained and corrected according to the state of the air, as indicated by the meteorological instruments. The subject of the astronomical refractions is thus resolved into two parts very distinct from one another; the first embracing the mean refractions, which are an unchangeable set of numbers, at least at every particular observatory; the second relating to the temporary variations occasioned by the fluctuations which are incessantly taking place in the condition of the atmosphere. It is the first of these two questions chiefly, or that regarding the mean refractions, of which it is proposed to treat in this paper.

In order to form a just notion of the mean refractions, we may suppose that some particular star is selected, and assiduously observed for a course of time so considerable as to comprehend every possible change in the condition of the atmosphere; all these observed places being severally reduced to some assumed state of the thermometer and barometer, and being combined so as to eliminate occasional irregularities, will determine the mean refraction of the star. In this procedure it is supposed, what experience confirms, that the result will ultimately be the same for the same altitude above the horizon, provided the observations are numerous enough, and extend over a sufficient length of time. We may instance the star α Lyræ observed by Dr. Brinkley; his observations are forty-four in number, extending over five years; and the greatest deviation of single observations from the mean quantity may be stated at $\pm 20''$. The supplementary Table, extending from $85°$ to $89°\frac{1}{2}$ of zenith distance, published in BESSEL'S *Tabulæ Regiomontanæ*, is one of mean refractions calculated from many

observations at every altitude. The Table, in the same work, extending to 85° of distance from the zenith, which the supplementary one is intended to complete, may likewise be considered as having the authority of actual observation; for although a theoretical formula was used in the calculations, yet the results have been carefully corrected by a comparison both with the observations of BRADLEY and with those made with very perfect instruments in the observatory over which BESSEL presides. These two make together a table of mean refractions of the highest authority; and being free from hypothetical admissions, to speak with precision, they form the only table of the kind of which astronomy in its actual state can boast.

The mean refractions, being a fixed set of numbers at any proposed observatory, are independent of temporary changes in the state of the air. If the general constitution of the atmosphere were so well known as to enable us to deduce the temperature, the density, and the pressure at any given altitude, from the observed condition of the air at the earth's surface, it might be possible to pitch upon an atmosphere intermediate between the extreme cases, in which the irregularities would compensate one another. From such an atmosphere the mean refractions used in astronomy might be correctly computed. But in reality we have no exact knowledge of the variations to which the air is subject in ascending above the surface of the earth. The diffusion of heat and aqueous vapour, the laws which regulate the density and pressure, are but slightly and hypothetically known. Many laborious researches in the lower part of the atmosphere, to which access can be had with instruments, have not been attended with complete success; and they have thrown no light upon what takes place in the upper parts. The limit of the atmosphere, or the height at which the air ceases to have power to refract light, is uncertain, and is no doubt, as well as the figure of the limiting surface, subject to continual fluctuation. Reflecting on what is said, it must be evident that the mean refraction of a star, which is a fixed quantity, cannot possibly be deduced from an atmosphere daily and hourly varying in its essential properties.

A table of refractions, such as is used in astronomy, contains only mean effects of the atmosphere, that take place at a given point of the earth's surface; and they should properly be compared with other mean effects at the same place. Of these mean effects a principal one is the height that must be ascended in the air for depressing the thermometer one degree, from which another mean effect is easily deduced, namely, the rate at which the density of the air decreases as the height increases. The values of these quantities, as occasionally determined at any particular place, will vary according to the actual state of the air; but a multitude of particular determinations embracing every vicissitude of the atmosphere, will at length lead to mean quantities which are constant, and such as would be observed in the same atmosphere that produces the mean refractions.

It is found that the refractive power of air depends on the density to which it is proportional; and hence the rate at which the density varies at the earth's surface

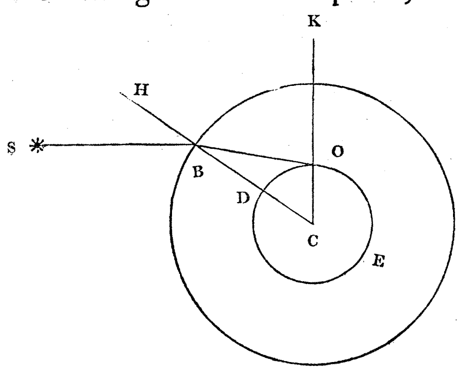
must have a great influence on the quantity of the astronomical refractions. It furnishes a key to the scale of the real densities in the atmosphere. When a thermometer is elevated in the air, it is found that the mercury continues to be depressed equably to great heights: in like manner the decrements of density will vary slowly from being proportional to the spaces passed through; so that a great share of that part of the astronomical refraction which depends upon the constitution of the atmosphere, must be ascribed to the initial rate at which the density decreases. This rate is not hypothetical; it is a real quantity independent of every other; its mean value, which alone we consider, is as determinate and as much the result of experiment as is the refractive power of the air: and in a solution of the problem which is not warped by arbitrary suppositions, and which deduces the effect only from causes really existing in nature, the former quantity will produce a part of the refraction as certain and unalterable, although perhaps not so considerable, as the latter.

But although the initial rate of the decrease of density is an essential element of the astronomical refractions, it may not alone be sufficient for a complete solution of the problem. In ascending to great heights above the earth's surface, the decrements of density will at length cease to be proportional to the spaces passed through, or to the variations of temperature. The refraction of light by the atmosphere is a complicated effect depending upon different considerations: but the influence of these considerations on the mean refractions must be uniform and free from fluctuation, and can arise only from quantities which are constant in their mean values at any proposed observatory. In speaking of mean quantities we exclude whatever is hypothetical, and confine our attention to such only as have a real existence in nature, although it may not in all cases be possible to obtain exact measurements by direct observation. As the refractions themselves are capable of being determined experimentally, they may be made the means of ascertaining what is left unknown in the formula for computing them; and they may thus contribute indirectly to advance our knowledge of the constitution of the atmosphere.

In proceeding to treat of this problem according to the notions that have been briefly explained, it remains to add that the mean effects of the atmosphere at the same observatory (of which mean effects a table of refractions is one) are alone considered, without at all entering on the question whether such effects are different or not, at different points of the earth's surface. It is very well known that the refractions, to a considerable distance from the zenith, depend only on the refractive power of the air and the spherical figure of the atmosphere: so far there is no reason to doubt that they are the same over a great part of the surface of the globe, according to the opinion generally held by astronomers: but, at greater zenith distances, when the manner in which the atmosphere is constituted comes into play, it is not so clear that they may not be subject to vary in different climates, and at different localities of the same climate. If a table of refractions at a given observatory contain a set of fixed numbers, these must be deducible from quantities not liable to change, that is,

from certain mean effects produced by the atmosphere at the observatory. To trace the relations that necessarily subsist between the mean effects that take place at a given point on the surface of the earth, is the proper business of geometry: if this can be successfully accomplished, the astronomical refractions will be made to depend upon a small number of quantities really existing in nature, and which can be determined, either directly or indirectly, by actual observation.

1. The foundation of the theory of the astronomical refractions was laid by DOMINIQUE CASSINI. The earth being supposed a perfect sphere, he conceived that it was environed by a spherical stratum of air uniform in its density from the bottom to the top. By these assumptions the computation of the refractions is reduced to a problem of the elementary geometry requiring only that there be known the height of the homogeneous atmosphere, and the refractive power of air. Let the light of a star S fall upon the atmosphere at B, from which point it is refracted to the eye of an observer at O on the earth's surface D O E: the centre of the earth being at C, draw the radii C O K, C D B H: the angle K O B $= \theta$, is the apparent zenith distance of the star; and O B C $= \varphi$ is the angle in which the light of the star is refracted on entering the atmosphere: now from the triangle O B C we deduce



$$\sin \mathrm{OBC} = \sin \mathrm{KOB} \times \frac{\mathrm{CO}}{\mathrm{CB}};$$

or, which is the same thing, putting $i = \frac{\mathrm{DB}}{\mathrm{CD}}$,

$$\sin \varphi = \frac{\sin \theta}{1 + i}.$$

Again, φ being the angle in which the light of the star is refracted, if we put $\delta \theta$ for the refraction, the angle of incidence S B H, which in the present case is always greater than the angle of refraction, will be $= \varphi + \delta \theta$; and $\frac{\sin (\varphi + \delta \theta)}{\sin \varphi}$ will be a constant ratio represented by $\frac{1}{\sqrt{1 - 2\alpha}}$; so that

$$\sin (\varphi + \delta \theta) = \frac{\sin \varphi}{\sqrt{1 - 2\alpha}} = \frac{\sin \theta}{(1 + i) \sqrt{1 - 2\alpha}}.$$

Thus we have the two following equations, which furnish a very easy rule for computing the mean refractions according to CASSINI's method, viz.

$$\sin \varphi = \frac{\sin . \theta}{1 + i}$$

$$\sin (\varphi + \delta \theta) = \frac{\sin \theta}{(1 + i) \sqrt{1 - 2\alpha}}.$$

As i and α are both very small numbers, if we put

$$m = i - i^2,$$

$$n = i - \alpha - i^2 + \alpha i - \frac{3\alpha^2}{2},$$

the two last equations will become

$$\sin \varphi = \sin \theta - m \sin \theta,$$

$$\sin (\varphi + \delta \theta) = \sin \theta - n \sin \theta:$$

and by employing the usual formula for deducing the variation of the arc from the variation of the sine, we get

$$\varphi = \theta - m \tan \theta + \frac{m^2}{2} \tan^3 \theta,$$

$$\varphi + \delta \theta = \theta - n \tan \theta + \frac{n^2}{2} \tan^3 \theta:$$

consequently

$$\delta \theta = (m - n) \tan \theta - \frac{m^2 - n^2}{2} \cdot \tan^3 \theta;$$

that is,

$$\delta \theta = \left(\alpha - i\alpha + \frac{3\alpha^2}{2}\right) \tan \theta - \left(i\alpha - \frac{\alpha^2}{2}\right) \tan^3 \theta;$$

or, which is the same thing,

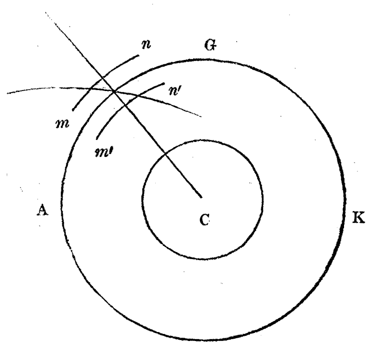
$$\delta \theta = \alpha \tan \theta \left(1 + \alpha - \frac{i - \frac{1}{2}\alpha}{\cos^2 \theta}\right) = \alpha (1 + \alpha) \tan \theta \left(1 - \frac{i - \frac{1}{2}\alpha}{\cos^2 \theta}\right),$$

agreeing exactly with LAPLACE's formula employed in computing the first part of the Table of mean refractions published by the French Board of Longitude.

2. The publication of NEWTON's Principia enabled geometers to take a more enlarged view of the astronomical refractions, and one approaching nearer to nature. According to CASSINI, the atmosphere is a spherical stratum of air, uniform in its density throughout, diffused round the earth to the height of about five miles; in reality the density decreases gradually in ascending, and is hardly so much attenuated as to be ineffective to refract the light at the great elevation of fifty miles. The path described by the light of a star in its passage through the atmosphere is therefore not a straight line, as it would be in the hypothesis of CASSINI, but a curve more and more inflected towards the earth's centre by the successive action of air of increasing density. Now in the Principia there is found whatever is necessary for determining the nature of this curve, and consequently for solving the problem of the astronomical refractions, which consists in ascertaining the difference between the direction of the light when it enters the atmosphere, and its ultimate direction when it arrives at the earth's surface. In the last section of the first book of his immortal work, NEWTON teaches in what manner the molecules of bodies act upon the rays of light and refract them; and as the atmosphere must be uniform in its condition at all equal altitudes, its action upon light can only be a force directed to the centre of the earth; so that

the trajectory in which the light moves, being described by a centripetal force, the determination of its figure will fall under the propositions contained in the second section of the same book.

Conceive that light falls upon an atmosphere A G K, constituted as CASSINI supposed, spherical in its form, concentric to the earth, of the same density ϱ throughout; and suppose that the attractive force of the molecules of air situated in the surface A G K extends to $m\ n$ on one side, and to $m'\ n'$ on the other. Every molecule of light when it arrives at $m\ n$ will be attracted by the air in a direction perpendicular to the surface A G K, and tending to C the centre of the earth; it will continue to suffer a varied attraction till it penetrates to the other surface $m'\ n'$; but when it has passed this limit, it will no longer be acted upon effectively by the surrounding air, which will attract it equally in all opposite directions. As the attraction of air extends only to insensible distances, in estimating its action upon a molecule of light we may consider the limiting surfaces $m\ n$ and $m'\ n'$ as parallel planes, the forces being perpendicular to $m\ n$, and of the same intensity at all equal distances from it. The law of the forces in action between $m\ n$ and $m'\ n'$ is indetermined; it may be uniform, or varied in any manner. These things being premised, it follows from a fundamental proposition of the philosophy of NEWTON, the demonstration of which it would be useless to repeat here, that the total action of all the forces between $m\ n$ and $m'\ n'$ is to add to the square of the velocity of the light incident at $m\ n$, an increment which is always the same, whatever be the direction in which the light arrives at $m\ n$. If we now put v for the velocity with which the light enters $m\ n$, and v' for the velocity with which it leaves $m'\ n'$, what is said will be expressed by this equation,



$$v'^2 - v^2 = 2 \,.\, \varphi\,(\varrho),$$

$\varphi\,(\varrho)$ denoting the sum of all the forces between $m\ n$ and $m'\ n'$, each multiplied by the space through which it acts, a sum which, in different atmospheres, will vary only when ϱ varies.

It will be convenient to have a name for the function $\varphi\,(\varrho)$, and the most appropriate term seems to be, the refractive power of the air. In using this term, or in expressing by $\varphi\,(\varrho)$ the action of air upon light, it is always supposed that the light passes out of a vacuum into air of the density ϱ.

A property resulting from what is said may be mentioned. Having drawn a radius from the centre of the earth to the point at which the light falls upon the atmosphere, let ϖ denote the angle made by the direction of the velocity v with the radius, and ϖ' the angle made by the direction of the velocity v' with it; then $v \sin \varpi$ and $v' \sin \varpi'$ will be the partial velocities of the light parallel to the surface of the atmosphere. Now these are equal; for all the forces which change v into v' are perpendicular to

the surface of the atmosphere, and therefore they have no effect to alter the velocity of the light parallel to that surface. Thus

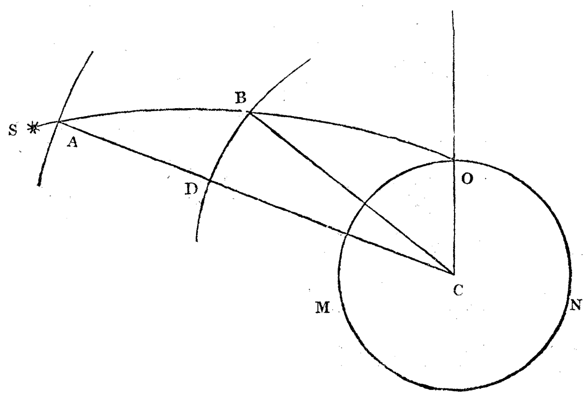
$$v \sin \varpi = v' \sin \varpi',$$

and

$$\frac{\sin \varpi}{\sin \varpi'} = \frac{v'}{v},$$

that is, in words, the ratio of the sine of incidence to the sine of refraction is equal to the ratio of the velocity of the light after refraction to the velocity of the incident light; which ratio, being independent of the direction of the incident light, is constant for all light that falls upon the atmosphere with the same velocity.

What has been said of an atmosphere supposed homogeneous is next to be applied to the real atmosphere of the earth, the density of which decreases continually in



ascending. The sphere M O N of which C is the centre, representing the earth, let S A B O be the trajectory described by light emanating from the star S, in its passage through the atmosphere to the earth's surface at O: through any two points of this curve, A and B, draw spherical surfaces concentric to the earth, the distances A C and D C from the common centre being $r + dr$ and r. Representing by ϱ the density of the air above the spherical surface at A, let $\varrho + d\varrho$ stand for the density, supposed uniform, of the stratum between the two surfaces at A and B: and it is to be observed that, though $AD = dr$ is an infinitesimal, it is nevertheless to be accounted infinitely great when compared to the insensible distance at which the molecular action of the air at A ceases to act: from which it follows that the refractive power of the stratum upon light which enters at A, is exactly equal to the refractive power of a homogeneous atmosphere, snpposing the density $\varrho + d\varrho$ to extend unvaried to the earth's surface. Now if v denote the velocity with which the light moves in the trajectory at A, the refractive power of the air above the stratum will diminish v^2 by the quantity $2 \varphi (\varrho)$; for it is obvious that the refractive power of the air above the spherical surface at A, is equal and opposite to the refractive power of a homogeneous atmosphere within the same surface and of the density ϱ: on the other hand the refractive power of the stratum will augment v^2 by the quantity $2 \varphi (\varrho + d\varrho)$:

wherefore, upon the whole, the real increment of v^2 will be $2\varphi(\varrho + d\varrho) - 2\varphi(\varrho)$; so that we shall have

$$d \, . \, v^2 = 2\varphi(\varrho + d\varrho) - 2\varphi(\varrho) = 2 \, d \, . \, \varphi(\varrho);$$

and, by integrating,

$$v^2 = n^2 + 2\varphi(\varrho).$$

It is obvious that n^2 is the square of the velocity of the light before it arrives at the atmosphere, that is, when it moves in a vacuum. We may consider n^2 as the unit in parts of which the squares of the velocity of the light at the several points of the trajectory are estimated; which requires that the formula be thus written,

$$v^2 = 1 + 2\varphi(\varrho).$$

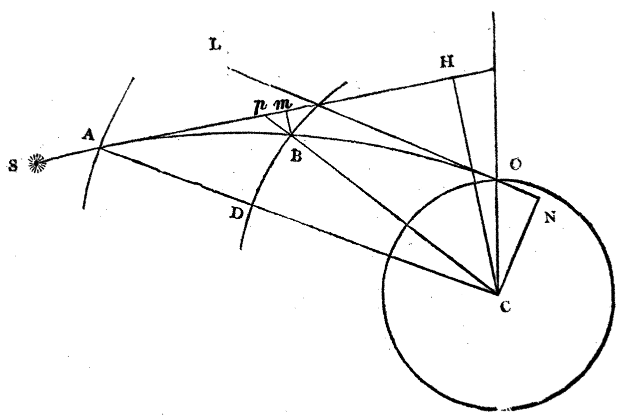
Resuming the equation

$$d \, . \, v^2 = 2 \, d \, . \, \varphi(\varrho),$$

we have

$$v \, d v = d \, . \, \varphi(\varrho) = \frac{d \, . \, \varphi(\varrho)}{d r} \, . \, d r;$$

from which we learn, that the same addition which v^2 receives by the refractive power of the air at A, it will acquire by the accumulated action of the force $\frac{d \, . \, \varphi(\rho)}{d r}$ at all the points of the line $d r$, or, which is the same thing, by the action of the force $\frac{d \, . \, \varphi(\varrho)}{d r}$ urging the light towards the earth's centre at all the points of the curve A B. Thus the path of the light of a star in its passage through the atmosphere is a trajectory described by the action of the centripetal force $- \frac{d \, . \, \varphi(\varrho)}{d r}$ tending to the centre of the earth, the sign $-$ being necessary, because the analytical expression is essentially negative.



Draw A H a tangent of the curve at A, B m perpendicular to A H, and produce C B to meet A H in p: put n for the angle A C O which the radius vector A C makes with C O the vertical of the observer; $d z$ for A B the element of the curve; $d \tau$ for the time of moving through A B; and R for the radius of curvature

at A. Now $\mathrm{B}p$ is the space through which the centripetal force $-\frac{d.\varphi(\varrho)}{dr}$ would cause a molecule of light to move from a state of rest in the time $d\tau$: wherefore

$$2\,\mathrm{B}p = -\frac{d.\varphi(\varrho)}{dr}\cdot d\tau^2;$$

also

$$\mathrm{B}p = \frac{\mathrm{B}m}{\sin \mathrm{BAD}} = \frac{dz^2}{2\mathrm{R}}\cdot\frac{dz}{r\,dn};$$

and, by equating the equal quantities, we get

$$\frac{dz}{\mathrm{R}} = -\frac{d.\varphi(\varrho)}{dr}\cdot\frac{d\tau^2}{dz^2}\cdot r\,dn. \quad \text{. . . . .} \quad (1.)$$

The refraction of the light in moving from A to B, or the difference of the directions of the curve at A and B, is evidently equal to the angle subtended by A B at the centre of the circle of curvature, that is, to $\frac{dz}{\mathrm{R}}$: wherefore if $\delta\,\theta$ represent the refraction increasing from the top of the atmosphere to the earth's surface, we shall have

$$d.\delta\theta = -\frac{d.\varphi(\varrho)}{dr}\cdot\frac{d\tau^2}{dz^2}\cdot r\,dn.$$

This formula is merely an application of the 6th proposition of the first book of the Principia.

Another general and useful expression of the differential of the refraction is easily obtained. Draw C H $= y$, perpendicular to the tangent A H: from the known properties of curve-lines, we have

$$\mathrm{R} = \frac{r\,dr}{dy}:$$

wherefore

$$\frac{dz}{\mathrm{R}} = dy\cdot\frac{dz}{dr}\cdot\frac{1}{r} = \frac{dy}{\sqrt{r^2-y^2}}:$$

consequently

$$d.\delta\theta = \frac{dy}{\sqrt{r^2-y^2}}, \quad \text{. . . . .} \quad (2.)$$

but in this formula $\delta\,\theta$ must be conceived to increase from the surface of the earth to the top of the atmosphere.

In applying the last formula it is necessary to have a value of y. Draw O L to touch the curve at O, and C N perpendicular to O L: put ϱ', v' for the density of the air, and the velocity of the light at O; also y' for the perpendicular C N, a for C O the radius of the earth, and θ for angle C O N, which is the apparent zenith-distance of the star: we shall have

$$\frac{\text{Area ABC}}{d\tau} = \frac{dz}{d\tau}\times y = v\times y:$$

and because the curve is described by a centripetal force tending to C, the value of

$v \times y$ will be the same at all the points of the curve; wherefore

$$v \times y = v' \times y';$$

and

$$y = y' \times \frac{v'}{v} = a \sin \theta \times \frac{v'}{v}.$$

Now, according to what was before shown,

$$v = \sqrt{1 + 2\varphi(\varrho)},$$
$$v' = \sqrt{1 + 2\varphi(\varrho')};$$

wherefore

$$y = a \sin \theta \times \sqrt{\frac{1 + 2\varphi(\varrho')}{1 + 2\varphi(\varrho)}}. \qquad (3.)$$

By substituting this expression in the differential of the refraction, the problem will be reduced to an integration.

The equations that have been investigated are perfectly general, and will apply in any constitution of the atmosphere that may be adopted. It has been thought better to consider the manner in which the forces act, than to employ functions with peculiar properties to express the molecular action. When the light in passing through the atmosphere arrives at a surface of increased density, it receives an impulse which may be considered as instantaneous; and this impulse being distributed over the breadth of a stratum of uniform density, ascertains the centripetal force tending to the earth's centre, by the action of which the trajectory is described.

3. It appears that NEWTON himself was the first to apply this new method to the problem of the astronomical refractions. A table, which he had computed, and which he gave to Dr. HALLEY, is published in the Philosophical Transactions for 1721. Nothing is said as to the manner in which the table was constructed: and it has always been a curious and interesting question among astronomers, whether it is the result of theory, or is deduced from observations alone, without the aid of theory. Some original letters of NEWTON to FLAMSTEED, published in 1835 at the expense of the Board of Admiralty, have put an end to all doubts on this point. These letters prove that NEWTON studied profoundly the problem of the refractions; that for several months in succession he was occupied almost entirely in repeated attempts to overcome the difficulties that occurred in this research; during which time he calculated several tables with great labour, namely, the one he gave to HALLEY, and another, or rather three others, which have been found in his letters to FLAMSTEED lately printed.

Admitting that the refractive power of the air is proportional to its density, which NEWTON had established in his Optics on speculative principles, and which HAWKSBEE afterwards was the first to demonstrate experimentally, the mathematical solution of the problem requires a knowledge of the law according to which the densities vary in the atmosphere. In his first attempt NEWTON assumes that the densities decrease in ascending in the same proportion that the distances from the earth's centre increase. Now a and r denoting the same things as before, put l for the total height

of the atmosphere; then $\varphi(\varrho)$ the refractive power of the air at the distance r from the centre of the earth, will, according to this hypothesis, be expressed by the formula

$$\varphi(\varrho) = \varphi(\varrho') \times \frac{l - r + a}{l}.$$

If this value be substituted in the formula (1.), which is a deduction from the sixth proposition of the first book of the Principia, the result will be

$$d \,.\, \delta\,\theta = \frac{\varphi(\rho')}{l} \cdot \frac{d\,\tau^2}{d\,z^2} \cdot r\,d\,n.$$

In this expression we have

$$\frac{d\,\tau^2}{d\,z^2} = \frac{1}{v^2} = \frac{1}{1 + 2\varphi(\rho)}:$$

and as $2\,\varphi(\varrho)$, or the increment of the square of the velocity of the light is very minute, amounting to less than ·0006 in passing through the whole atmosphere to the earth's surface, we may reckon $\frac{d\,\tau^2}{d\,z^2}$ as unit; thus we get

$$d \,.\, \delta\,\theta = \frac{\varphi(\rho')}{l} \cdot r\,d\,n;$$

and by integrating

$$\delta\,\theta = \frac{2\,\varphi(\rho')}{\left(\frac{l}{a}\right)} \cdot \int \frac{r\,a\,d\,n}{2\,a^2}.$$

This result, which M. Biot has also obtained, is equivalent to the geometrical construction communicated by Newton to Flamsteed in a letter from Cambridge, December 20, 1694. The problem was now reduced to the quadrature of a curve, for which a general method is given in the fifth lemma of the third book of the Principia, a method which is still used when the direct process of integration fails, or becomes too intricate for practice. What has been said not only proves the exactness of Newton's solution of the problem; it also points out, with little uncertainty, the manner in which he obtained it. Of the arithmetical operations of the quadrature there is no account; and they would be of no interest had they been preserved. He complains much of the great labour of the numerical calculations; but all difficulties were overcome, as was to be expected: a table was computed and communicated to Flamsteed in a triple form, for summer, winter, and the intermediate seasons of spring and autumn. On mature reflection there occurred to him a serious objection to the supposed scale of densities, on which account he writes to Flamsteed that he does not intend to publish the tables. The fault lies in this, that the centripetal force which continually inflects the light to the earth's centre, is the same at all the points of the trajectory, or, in the words of Newton, the refractive power of the atmosphere is as great at the top as at the bottom,—than which nothing certainly can be more different from what actually takes place in nature.

Dismissing his first hypothesis, NEWTON next turned his attention to the 22nd proposition of the third book of his Principia. If the atmosphere consist of an elastic fluid gravitating to the earth's centre in the inverse proportion of the square of the distance, and if it be admitted that the densities are proportional to the pressures, NEWTON, in the proposition cited, proves in effect that the densities will form a decreasing geometrical series, when the altitudes are taken in arithmetical progression*. He writes to FLAMSTEED that an atmosphere so constituted is *certainly the truth.* NEWTON evidently intended by this assertion to mark a distinction between pressure, which is a cause of the variation of density that actually exists in nature, and his first assumed law of the densities, which is entirely arbitrary. Setting aside hypothesis, he now advanced so far in the true path of investigation; and if the manner in which heat is diffused in the atmosphere and the consequent decrease of density were not known when he wrote, he advanced as far as the existing state of knowledge enabled him to do. It is certain from his letters, that, after much time and labour, he at last succeeded in calculating a table of refractions on the principle that the density is proportional to the pressure. Such a table he communicated to FLAMSTEED, although it is not found in the letters lately published; and there is every reason to think it the same which he gave to HALLEY, and which that astronomer inserted in the Philosophical Transactions for 1721. Two elements only are sufficient for computing all the numbers in a table of refractions constructed by assuming that the density is proportional to the pressure, namely, the refraction at 45° of altitude, and the height of the homogeneous atmosphere, which is deducible from the horizontal refraction. The table of HALLEY, therefore, contains in itself all that is required for ascertaining whether it was calculated or not by the principle alluded to in the letters of NEWTON to FLAMSTEED. KRAMP seems to be the first who sought in the table for the manner of its construction; and his discoveries in this branch of science enabled him to assign the height of the homogeneous atmosphere, which is one essential element. The refraction at 45° of altitude, which is the other element, is found in the table equal to 54″, or, in parts of the radius, to ·0002618; and KRAMP found 4377† toises for l, the height of the homogeneous atmosphere; so that, if a be the radius of the earth in toises, we have

$$\alpha = \cdot 0002618,$$

$$i = \frac{l}{a} = \cdot 0013356;$$

and the two elements, α and i, are sufficient for computing the whole table, if it be

* NEWTON demonstrates strictly that the densities will be in geometrical proportion when the distances from the earth's centre are in musical or harmonical proportion, that is, when they are the reciprocals of an arithmetical progression; but in a series of this kind, if the first term bear an almost infinitely great proportion to the differences of the following terms, as is the case of the radius of the earth when compared to elevations within the limits of the atmosphere, the differences of the terms or the elevations may, without sensible error, be reckoned in arithmetical progression.

† Anal. des Réfractions Astronomiques, p. 19.

such as is mentioned in the correspondence between NEWTON and FLAMSTEED. The formula for the refraction in the supposed constitution of the atmosphere has been given both by KRAMP and LAPLACE; and it may be taken from the Paper in the Philosophical Transactions for 1823, p. 441,

$$\lambda = \frac{\alpha}{i} = \cdot 19601, \quad \Delta = \sqrt{\cos^2\theta + 2\,i\,s},$$

$$\begin{aligned} \delta\theta = \alpha \sin\theta \times \Big\{ & \left(1 - \lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{6}\right) . \int \frac{d\,s\,c^{-s}}{\Delta} \\ & + (\lambda - 2\lambda^2 + 2\lambda^3) . \int \frac{2\,d\,s\,c^{-2s}}{\Delta} \\ & + \left(\frac{3}{2}\lambda^2 - \frac{9}{2}\lambda^3\right) . \int \frac{3\,d\,s\,c^{-3s}}{\Delta} \\ & + \frac{8\lambda^3}{3} . \int \frac{4\,d\,s\,c^{-4s}}{\Delta}, \end{aligned}$$

the integrations extending from $s = 0$ to $s = \infty$. The coefficients of this formula are as follows:

$$A = 1 - \lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{6} = \cdot 82193$$

$$B = \lambda - 2\lambda^2 + 2\lambda^3 = \cdot 13423$$

$$C = \frac{3}{2}\lambda^2 - \frac{9}{2}\lambda^3 = \cdot 02377.$$

$$D = \frac{8}{3}\lambda^3 = \cdot 02007.$$

For the horizontal refraction, when $\cos\theta = 0$, $\Delta = \sqrt{2\,i\,s}$, we obtain by the usual integrations,

$$\delta\theta = \frac{\alpha\sqrt{\varpi}}{\sqrt{2\,i}} \times \left\{A + B\sqrt{2} + C\sqrt{3} + D\sqrt{4}\right\}:$$

or in seconds, $\delta\theta = 2024 \cdot 2$ instead of $2025''$ as in HALLEY's table. This proves the exactness of KRAMP's elements.

With respect to the other numbers in the table a distinction must be made. In every table of refractions, whatever be the constitution of the atmosphere on which it is founded, the numbers answering to altitudes greater than 16°, depend only upon one element, namely, the refractive power of the air. Reckoning from the zenith as far as 74°, any table may be deduced from any other, provided both are accurately calculated, merely by a proportion. In the table published annually in the *Con. des Temps*, the refraction at 45° is $58''\cdot 2$: and, if HALLEY's table has been accurately computed, the numbers in it, between the limits mentioned, will be equal to the like numbers in the French table multiplied by $\frac{540}{582} = \frac{90}{97}$. The calculation being made,

the results will be found to agree almost exactly with the short table inserted by M. BIOT in the additions to the *Con. des Temps* for 1839, p. 105, the greatest difference between the computed quantities and the numbers in HALLEY's table being about $2''$.

But this gives no intimation with respect to the particular constitution of the atmosphere assumed in the calculation of the table. What is peculiar to a table in this respect has no sensible influence on the refractions it contains except at altitudes less than 16°. No definitive opinion can therefore be formed on the question, whether HALLEY's table is the same which NEWTON computed and communicated to FLAMSTEED on the principle that the densities are proportional to the pressures, without comparing it with a sufficient number of refractions at low altitudes calculated from the elements of KRAMP. As the settling of this point may be thought not unimportant, the following formula, which affords the means of computing the refractions at all altitudes with exactness, has been investigated by reducing the integrals in the expression of $\delta\,\theta$ to serieses.

$$\operatorname{Tan}\varphi = \frac{2\sqrt{5i}}{\cos\theta}, \quad e = \tan\frac{\varphi}{2}.$$

$$\operatorname{Log}\tan\varphi = \log \operatorname{secant}\theta + 19{\cdot}2133569 - 20.$$

$$\delta\,\theta = \sin\theta \times \left\{ \begin{array}{llrl} & e & \times\ 660''{\cdot}795 \ldots\ldots & \overset{\log.}{2{\cdot}8200669} \\ + & e^3 & \times\ 551{\cdot}634 \ldots\ldots & 2{\cdot}7337059 \\ + & e^5 & \times\ 371{\cdot}268 \ldots\ldots & 2{\cdot}5696873 \\ + & e^7 & \times\ 219{\cdot}762 \ldots\ldots & 2{\cdot}3419630 \\ + & e^9 & \times\ 116{\cdot}763 \ldots\ldots & 2{\cdot}0673034 \\ + & e^{11} & \times\ 58{\cdot}170 \ldots\ldots & 1{\cdot}7646976 \\ + & e^{13} & \times\ 28{\cdot}275 \ldots\ldots & 1{\cdot}4514092 \\ + & e^{15} & \times\ 13{\cdot}797 \ldots\ldots & 1{\cdot}1397974 \\ + & e^{17} & \times\ 6{\cdot}806 \ldots\ldots & 0{\cdot}8329041 \\ + & e^{19} & \times\ 3{\cdot}311 \ldots\ldots & 0{\cdot}5199046 \end{array} \right\}$$

The series converges very slowly, which has made it necessary to continue it to ten terms, the amount of which is still $3''{\cdot}6$ deficient from the exact quantity $2024''{\cdot}2$. As the last terms decrease in the proportion of 2 to 1, it is obvious that the true sum would be obtained by continuing the series: but the terms set down are more than sufficient for the present purpose.

The exact refractions calculated by the formula are next to be compared with the numbers in HALLEY's table.

Apparent zenith-dist.	Refractions		Difference.
	Computed.	HALLEY's Table.	
°	′ ″	′ ″	″
20	19·6	20	−0·4
40	45·2	45	+0·2
60	1 33·1	1 32	+1·1
70	2 27·0	2 26	+1·0
80	4 55·2	4 52	+3·2
82	6 3·9	6 00	+3·9
84	7 45·0	7 45	0·0
86	10 53·1	10 48	+5·1
87	13 22·5	13 20	+2·5
88	17 4·6	17 8	−3·4
89	23 3·5	23 7	−3·5

The examination that has now been made fully establishes that HALLEY's table is no other than the one which NEWTON computed on the supposition that the densities in the atmosphere are proportional to the pressures. The discrepancies, amounting in every instance to a small part of the whole quantity, are such as might be expected to occur unavoidably in the intricate and laborious methods of calculation followed by NEWTON. Some part of the differences may also arise from the elements of the formula, which, being deduced from only two numbers of the table, may not exactly coincide with the fundamental quantities which NEWTON assumed. M. BIOT, by a method different from KRAMP's, has found other elements, which make the horizontal refraction 8″·3 less than in the table; but these small variations, which proceed from calculating in different ways, only confirm more strongly that HALLEY's table is the same which NEWTON constructed by his second hypothesis, and communicated to FLAMSTEED.

It appears from what has been said, that, as far as the mathematics is concerned, the problem of the astronomical refractions was fully mastered by NEWTON. It would be strange indeed to suppose that the author of the theories in the Principia would find difficulty to apply them in a case to which he bent the whole force of his mind. But he was embarrassed by the suppositions he found it necessary to make respecting the physical constitution of the atmosphere. His first hypothesis is evidently contrary to nature, in admitting that air at all altitudes exerts the same power to inflect the light to the earth's centre; his second method made the refractions too great near the horizon, thereby proving the necessity of searching out some new cause for the purpose of reconciling the theory with observation. Averse from hypotheses, he seems, on these accounts, to have declined inserting in his works a problem which had cost him so much labour, and upon his solution of which he evidently set some value.

If the different attempts to solve this problem are to be tried with the same rigour

that NEWTON judged his own, it must be decided that they are all liable to objections. They all involve some supposition that has no foundation in nature; or they leave out some necessary condition of the problem. It is allowed that the variation of heat at different altitudes is unknown; that we are equally unacquainted with the manner in which is diffused the aqueous vapour that is always found, more or less, in the atmosphere; that the law of the densities has not been ascertained. But besides these capital points, the accurate M. POISSON will suggest other properties that must be attended to in an atmosphere in equilibrium: the conducting power of heat varying with the condition of the air; its power of absorbing heat; and the interchange by radiation which takes place with the earth, with the etherial spaces above, and with the stars visible above the horizon. So many conditions, placed beyond the reach of our inquiry, may well puzzle the most expert algebraist to take them into account. But it may be doubted whether this be the proper view of the problem. The astronomical refractions at any observatory are mean effects of the atmosphere; and it may be alleged that the proper way of accounting for them is to compare them with other mean effects of the atmosphere at the same place.

4. In 1715, twenty years after the researches of NEWTON, BROOK TAYLOR published in his *Methodus Incrementorum*, the first investigation of the astronomical refractions on the supposition that the density of the air is variable. The differential equation is accurately deduced from the principles laid down by NEWTON; which removed all the difficulties of the problem in this respect.

KRAMP, in his *Analyse des Réfractions Astronomiques et Terrestres*, has elucidated the elementary parts of the problem, and has greatly improved the mathematical solution. His method is particularly commodious and useful in the case of the horizontal refraction. For altitudes above the horizon the integrals are not susceptible of being simply expressed, and seem to require the aid of subsidiary tables in applying them.

The problem of the refractions being an important one in astronomy, many solutions of it have been published by different geometers. Some of these are preferable to others, because the method of calculation is easier in practice. For altitudes greater than 16°, they may all be reckoned equivalent, differing from one another only in the quantity assumed for the refractive power of air. They also mostly agree in the horizontal refraction, which is taken from observation. But for altitudes less than 16°, they are different: because, at these low altitudes, the refractions are affected by the arbitrary suppositions used in constructing the tables.

THOMAS SIMPSON published a judicious dissertation on this problem. He distinctly points out that, to a considerable distance from the zenith, the refractions are independent of the manner in which the atmosphere is supposed to be constituted. In comparing the two atmospheres that have densities decreasing in arithmetical and geometrical progression, he remarks that the horizontal refraction comes much nearer the observed quantity in the first atmosphere than it does in the second; for which

reason he gives the preference to the first as likely to represent the phenomena with greater accuracy. Now in this his reasoning is not much different from the argument afterwards used by LAPLACE, to prove that the same two atmospheres are limits between which the true atmosphere is contained.

NEWTON likewise found that the refractions computed according to his second method, that is, in an atmosphere with densities decreasing in geometrical progression, are too great near the horizon, on which point he thus writes to FLAMSTEED. "Supposing the atmosphere to be constituted in the manner described in the 22nd proposition of my second book (which certainly is the truth), I have found that if the horizontal refraction be 34′, the refraction at the altitude of 3° will be 13′ 3″; and if the refraction in the apparent altitude of 3° be 14′, the horizontal refraction will be something more than 37′. So that instead of increasing the horizontal refraction by vapours, we must find some other cause to decrease it. And I cannot think of any other cause besides the rarefaction of the lower region by heat." Here the true reason is assigned why the refractions near the horizon, in an atmosphere constituted as supposed, so much exceed the observed quantities. When the density is made to depend solely on the incumbent weight, the air is not rarefied enough; and the greater density causes a greater refraction. Having correctly estimated the effect produced by the pressure of the supported air, NEWTON is unavoidably led to ascribe to heat the greater rarefaction that takes place in the atmosphere of nature. His words prove that he had no clear conception in what manner the density in the lower region is altered by the agency of heat; and, to say the truth, nearly the same ignorance in this respect prevails now as in his time. The decrease of density in ascending is a complicated effect of many causes for the most part unknown; and it seems in vain to expect a satisfactory investigation of it by arbitrary suppositions. But setting aside hypothetical constitutions of the atmosphere, we may consider the rarefaction of the air in ascending as a phenomenon, the knowledge of which is to be acquired by experiment; and this appears the only sure way of placing the theory of the mean refractions on its proper foundation.

5. One of the tables of refraction most esteemed by astronomers is that published annually in the *Con. des Temps.* It has been already shown that, as far as 74° from the zenith, this table is calculated by the simple method of CASSINI. There is nothing incidental in this; for all tables of refraction may be computed by CASSINI's method to the extent mentioned. The French astronomers have been very successful in determining the constants of the formula. The refractive power of the air was obtained by DELAMBRE from a great number of astronomical observations; the same quantity was deduced by MM. BIOT and ARAGO from experiments on the gases with the prism; and the results of two methods, so entirely different, agree so nearly, that there seems no ground for preferring one to the other. The remaining part of the French table, for altitudes less than 16°, is computed by a method of LAPLACE, which the author has explained, without disguising its defects, in the fourth volume of the *Mec. Celeste.*

The two atmospheres with densities decreasing in arithmetical and geometrical progression, which it now appears were imagined by NEWTON, and which have been discussed by THOMAS SIMPSON and other geometers, are found, when the same elements are employed, to bring out horizontal refractions on opposite sides of the observed quantity. LAPLACE conjectured that an intermediate atmosphere which should partake of the nature of both, and should agree with observation in the horizontal refraction, would approach nearly to the true atmosphere. It must be allowed that these conditions, which may be verified by innumerable instances between the two limits, are vaguely defined; and in order to ascertain the real meaning of the author, recourse must be had to the algebraic expressions. When this is done it will be found that the atmosphere intended is one of which the density is the product of two terms, one taken from an arithmetical, and the other from a geometrical progression; the effect of which combination is to introduce a supernumerary constant, by means of which the horizontal refraction is made to agree with the true quantity. No one will deny the merit and the ingenuity of LAPLACE's procedure; but though very skilful, and guided in some degree by fact, it is liable to all the uncertainty of other arbitrary suppositions, as indeed the author allows. Dr. BRINKLEY has given the character of the French table fairly when he says, that it is only a little less empirical than the other tables. On divesting LAPLACE's hypothesis of vagueness in the language, and expressing it in the unequivocal symbols of algebra, it does not appear to possess any superiority over other supposed constitutions of the atmosphere in leading to a better and less exceptionable theory; at least the *Mec. Celeste* has been many years before the public, during which time not a few geometers have laboured on the subject of the refractions; but no improvement originating in the speculations peculiar to LAPLACE has occurred to any of them.

Having said so much on the theory of the French table, it may be proper to add a word on its accuracy. If it be compared from 80° to 88° of zenith distance with BESSEL's observed refractions, there will be found a small error in excess, continually increasing, and amounting at last to 4″. This shows that, in combining the two atmospheres, too much weight has been given to that with a density varying in geometrical progression, in consequence of which the air is not rarefied enough in the interpolated atmosphere. With respect to the two degrees of altitude next the horizon, no accurate judgement can be formed, for want of observed refractions that can be depended on.

The astronomical refractions have also occupied the attention of the astronomer of Kœnigsberg, who has contributed so largely to the improvement of every part of astronomical science. For the purpose of representing the observations of BRADLEY with all the accuracy possible, BESSEL investigated a table of refractions which appeared in the *Fundamenta Astronomiæ* in 1818. He assumes a theoretical formula; but as every arbitrary quantity is determined by a careful comparison with real observations, what is supposititious may be considered merely as an instrument of investigation, which is finally laid aside, leaving the result to rest on the foundation of

fact. He returned to the subject in his *Tabulæ Regiomontanæ*, published in 1830. In this last work he retains only that part of the table of 1818 which extends to 85° from the zenith, many corrections being applied from recent observations made with improved instruments. In order to supply what is wanting in the new table, BESSEL has added a supplemental one containing the refractions at every half degree for altitudes less than 15°: which supplemental table is independent of theory, being deduced from observations alone. These two tables form together a real table of mean refractions, independent of all suppositions respecting the constitution of the atmosphere; and no other similar table of nearly equal authority is to be found in the astronomy of the present day. What BESSEL has accomplished on the subject of the refractions is not the least important part of his labours for the advancement of astronomical science: it is precious to the practical astronomer; and it is necessary to the theoretical inquirer, for enabling him to confront his speculations with the phenomena to be accounted for.

6. In the paper published in the Philosophical Transactions for 1823, the refractions are deduced entirely from this very simple formula,

$$\frac{1+\beta\tau}{1+\beta\tau'} = 1 - f(1 - c^{-u}), \quad \ldots \quad (4.)$$

in which β stands for the dilatation of air, or a gas, by heat; τ' is the temperature at the earth's surface, and τ the temperature at any height above the earth's surface; at the same height c^{-u} is the density of the air in parts of its density at the surface.

In order to understand the application of the formula, it is necessary to premise that in the remaining part of this paper we do not consider a variable atmosphere subject to continual fluctuations, as is the case of the real atmosphere: we contemplate an atmosphere fixed in its condition at any given place or observatory, being supposed a mean between all the variations that actually take place in an indefinite time. In such an atmosphere the temperature and pressure at the earth's surface will be mean quantities deduced from observation: the air at all elevations will have an elastic force equal to the incumbent weight which it supports, as an equilibrium requires: and, whether the air be dry or moist, its refractive power will be equal to the refractive power of dry air subjected to the same pressure and temperature*. These properties of the mean atmosphere rest upon experiment and demonstration: in other respects its nature is not directly known to us: and the laws of its action can only be discovered, not by hypothesis, but by observation.

The consideration of a mean atmosphere, invariable at any given observatory, is a necessary consequence of the notion we attach to the mean refractions; for these would be realized in such an atmosphere: but they are different in any other state of the air.

These observations being premised, if the formula (4.) be verified at the earth's

* Additions à la Conn. des Temps, 1839, p. 36.

surface in any invariable atmosphere, by giving a proper value to the constant f, it will still hold, at least with a very small deviation from exactness, at a great elevation, probably at a greater elevation than has ever been reached by man. In order to prove this, let the arbitrary function $\varphi\,(u)$ be added, so as to complete the formula by rendering it perfectly exact: then

$$\frac{1+\beta\tau}{1+\beta\tau'} = 1 - f(1-c^{-u}) - \varphi\,(u), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

and it will follow that $\varphi\,(u) = 0$, when $u = 0$, that is, at the earth's surface. Again, differentiate the equation, observing that τ decreases when u increases, then

$$\frac{\beta}{1+\beta\tau'} \cdot \frac{d\tau}{du} = f c^{-u} + \frac{d\,.\,\varphi\,(u)}{du}:$$

now, since this equation is true for all values of u, it will hold at the earth's surface, or when $u = 0$: and if f be taken equal to the particular value of

$$\frac{\beta}{1+\beta\tau'} \cdot \frac{du}{d\tau},$$

when $u = 0$, it will follow that $\frac{d\,.\,\varphi\,(u)}{du} = 0$, when $u = 0$. And since the equations $\varphi\,(u) = 0$ and $\frac{d\,.\,\varphi\,(u)}{du} = 0$, are both verified at the earth's surface, it follows that the supplementary function $\varphi\,(u)$ will vary slowly as u increases, that is, as the density of the air decreases in ascending. This proves that the approximate equation (4) will be very little different from the exact equation (5.) at great elevations in the atmosphere.

At the surface of the earth du is equal to the variation of the density for a depression of the thermometer expressed by $d\tau$: and although the proportion of these two quantities cannot be ascertained by direct experiment, yet, as is shown in the paper of 1823, it is easily deduced from the rate at which the temperature decreases as the height increases, which rate is easily determined experimentally. The quantity f thus found is necessarily constant at the same observatory. It is the mean of all the occasional values, which vary incessantly, while the real atmosphere undergoes every vicissitude of which it is susceptible. The mean refraction and f are invariable in quantity, because they depend alike upon the mean condition of the air at a given place. Some confusion has arisen on this point from not distinguishing between the mean refraction of a star and its true refraction in a variable atmosphere.

In all that has been said there is no supposition of an arbitrary constitution of the atmosphere. The assumed formula (4.) is an approximate truth in every invariable state of the air in equilibrium. Conceive a cylindrical column of air perpendicular to the earth's surface, and extending to the top of the atmosphere; at the height where the temperature is τ, and the density ϱ, let p denote the weight of the column above the height; and suppose that p, ϱ, τ are changed into p', ϱ', τ' at the surface of the

earth; because the pressures are proportional to the elasticities, we have the usual equation,

$$\frac{p}{p'} = \frac{1 + \beta\tau}{1 + \beta\tau'} \times \frac{\rho}{\rho'}:$$

or, which is the same,

$$c^{-u} = \frac{\rho}{\rho'}, \ \frac{p}{p'} = \frac{1 + \beta\tau}{1 + \beta\tau'} \times c^{-u}:$$

and by substituting the complete expression of the temperature as given in (5.), we shall obtain,

$$\frac{p}{p'} = c^{-u} - f(c^{-u} - c^{-2u}) - c^{-u} \times \phi(u): \quad . \quad . \quad . \quad . \quad (6.)$$

and if we omit the supplemental part, which is small even at great elevations, the result will be,

$$\frac{p}{p'} = c^{-u} - f(c^{-u} - c^{-2u}).$$

Now this is the constitution of the atmosphere in the paper of 1823; it is only approximate; but it is an approximation applicable to every atmosphere that can be imagined, requiring no more than that the quantity f have the proper experimental value given to it. It is shown in the paper that the pressures and densities are thus represented with no small degree of accuracy at the greatest heights that have been reached; which proves how slowly the supplemental part of the formula (5.) comes into play.

The foregoing manner of arriving at the constitution of the atmosphere adopted in the paper of 1823, being drawn from properties immediately applicable to the problem in hand, is more natural, and more likely to suggest itself, and more satisfactory than the ingenious and far-fetched procedure of M. Biot, of transforming an algebraic formula for the express purpose of bringing out a given result. Laplace, having remarked that the true horizontal refraction is contained between the like quantities of two atmospheres, with densities decreasing in arithmetical and geometrical progression, conjectured that an atmosphere between the two limits, which should likewise agree with observation at the horizon, would in all probability represent the mean refractions with considerable accuracy. It is upon this assumption that the problem is solved in the *Mec. Celeste*, the observed horizontal refraction being used for determining the arbitrary constant. Now in the paper of 1823 there is no allusion to interpolating an atmosphere between two others; a knowledge of the horizontal refraction is not required; the investigation is grounded upon a property common to every atmosphere in a quiescent state; and lastly, the resulting table is essentially different from all the tables computed by other methods. If these considerations be not sufficient to stamp an appropriate character upon the solution of a problem, it would be difficult to find out what will be sufficient. But if it be possible, with M. Biot's ingenuity, to trace some relation in respect of the algebraic

expressions, between the paper of 1823 and the calculations of LAPLACE, from which, after all, no just inference can be drawn, it is not difficult to find between the same paper and the view of the problem taken by the author of the Principia, in 1696, an anology much more simple and striking, which deserves to be mentioned as it tends to bring back the investigation to the right tract, which it seems to have left. NEWTON, having solved the problem on the supposition that the density of the air is produced solely by pressure, found that the refractions thus obtained greatly exceeded the observed quantities near the horizon: and hence he inferred, in the true spirit of research, that there must be some cause not taken into account, such as the agency of heat, which should produce, in the lower part of the atmosphere, the proper degree of rarefaction necessary to reconcile the theoretical with the observed refractions. Now, in the paper of 1823, the sole intention of introducing the quantity f, not noticed before by any geometer, is to cause the heat at the earth's surface to decrease in ascending at the same rate that actually prevails in nature; which evidently has the effect of supplying the desideratum of NEWTON.

The remarks that have just been made are not called for by anything which M. BIOT has written in his dissertation on the refractions, inserted in the additions to the *Conn. des Temps* for 1839; because that author has fully explained the grounds of what he advances, thereby enabling a candid inquirer to form his own opinion: but all the world are not of the same character as that distinguished philosopher.

At every point on the earth's surface we are now acquainted with three things, not hypothetical or precarious, that have an influence on the mean refractions. These are, the refractive power of the air, the spherical figure of the atmosphere, and the mean rate at which the density of the air decreases at the given place. These three things are independent on one another, and on all other properties of the air: they will therefore produce three independent parts of the quantity sought. The parts thus determined may fall short of the whole refraction at any altitude, because there may be causes not taken into account that co-operate in producing the result: but each will unalterably maintain its proper share of the total amount, in whatever way it is attempted to solve the problem, provided the solution is conducted on right principles and not warped by arbitrary suppositions. It may therefore be said that, in so far, an advance has been made in acquiring an exact notion of the nature of this problem.

The table in the paper of 1823 was compared with the best observations that could be procured at the time of publication; and the results were very satisfactory. After the publication of the *Tabulæ Regiomontanæ*, it was found that the table agreed with BESSEL's observed refractions to the distance of 88° from the zenith, which is as far as his determinations can be depended on, with such small discrepancies as may be supposed to exist in the observations themselves. So close an agreement between the theoretical and observed mean refractions was very unexpected, and even contrary to the opinion very generally held on this subject.

Astronomers are in the habit of using different tables or formulas of refraction, which, being derived from conjectural views, do not agree with one another, except to a limited distance from the zenith. Now this is contrary to the very conception we have of the mean refractions, which are determinate and invariable numbers, at least at the same observatory. A great advantage would therefore ensue from setting aside every uncertain table, and substituting in its place one deduced from the causes really existing in nature that produce the phenomena. Such a table adapted to every observatory, if this were found necessary, would contribute to the advancement of astronomy by rendering the observations made at different places more accurately comparable. It might contribute to the advancement of knowledge in another respect: for if the mean refractions were accurately settled, the uncertainty in the place of a star would fall upon the occasional corrections depending on the indications of the meteorological instruments; and it is not unreasonable to expect that much which is at present obscure and perplexing on this head might be cleared up, if it were separated from all foreign irregularities, and made the subject of the undivided attention of observers.

7. The paper in the Philosophical Transactions for 1823 takes into account only the rate at which the densities in a mean atmosphere vary at the surface of the earth; what follows is an attempt to complete the solution of the problem by estimating the effect of all the quantities on which the density at any height depends. For this purpose it will be requisite to employ certain functions of a particular kind, viz.

$$\mathrm{R}_1 = 1 - c^{-u},$$

$$\mathrm{R}_2 = 1 - u - c^{-u},$$

$$\mathrm{R}_3 = 1 - u + \frac{u^2}{1.2} - c^{-u},$$

$$\vdots$$

$$\mathrm{R}_i = \left(1 - u + \frac{u^2}{1.2} - \frac{u^3}{1.2.3} \cdots \pm \frac{u^{i-1}}{1.2.3\ldots i-1}\right) - c^{-u}.$$

In these expressions c is the number of which the hyperbolic logarithm is unit; and it is obvious that R_i is zero when $u = 0$. These expressions have several remarkable properties, which are proved by merely performing the operations indicated.

1st. $$\frac{d.\mathrm{R}_i}{du} = -\mathrm{R}_{i-1},$$

$$\int -\mathrm{R}_i\, du = \mathrm{R}_{i+1},$$

the integral being taken equal to zero, when $u = 0$.

2ndly. $$\frac{d.c^{-u}\mathrm{R}_i}{c^{-u}.du} = -(\mathrm{R}_{i-1} + \mathrm{R}_i),$$

$$\frac{d\,d \,.\, c^{-u}\,R_i}{c^{-u} \,.\, d\,u^2} = R_{i-2} + 2\,R_{i-1} + R_i$$

$$\frac{d^3 \,.\, c^{-u}\,R_i}{c^{-u}\,d\,u^3} = -\,(R_{i-3} + 3\,R_{i-2} + 3\,R_{i-1} + R_i),$$

&c.

3rdly. n being less than i,

$$\int \frac{d^n \,.\, c^{-u}\,R_i}{c^{-u} \,.\, d\,u^n}\,d\,u = (-\,1)^n \cdot \frac{d^n \,.\, c^{-u}\,R_{i+1}}{c^{-u} \,.\, d\,u^n}.$$

These things being premised, the temperature of an atmosphere in equilibrium will have for its complete expression this formula,

$$\frac{1+\beta\,\tau}{1+\beta\,\tau'} = 1 - f\,R_1 - f'\,\frac{d \,.\, c^{-u}\,R_3}{c^{-u} \,.\, d\,u} - f''\cdot\frac{d^2 \,.\, c^{-u}\,R_5}{c^{-u} \,.\, d\,u^2} - \text{\&c.} \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

the coefficients f, f', f'', &c. being indeterminate constant quantities. A little attention will show that this expression is equivalent to a series of the powers of u; for, first, let the differential operations in the several terms be performed, which will bring out

$$\frac{1+\beta\,\tau}{1+\beta\,\tau'} = 1 - f\,R_1 + f'\,(R_2 + R_3) - f''\,(R_3 + 2\,R_4 + R_5) + \text{\&c.};$$

next, expand R_1, R_2, &c., and the result will be,

$$\begin{aligned}\frac{1+\beta\,\tau}{1+\beta\,\tau'} = 1 - f\,u &+ (f - f')\cdot\frac{u^2}{1\,.\,2} \\ &- (f - 2f' + f'')\cdot\frac{u^3}{1\,.\,2\,.\,3} \\ &+ (f - 2f' + 3f'' - f''')\,\frac{u^4}{1\,.\,2\,.\,3\,.\,4} \\ &- \text{\&c.}\end{aligned}$$

The intention of assuming the formula (7.) is to express the temperature in terms of such a form as will produce, in the refraction, independent parts that decrease rapidly.

In order to elucidate what is said, and more especially to prove that the analysis here followed comprehends all atmospheres, whether of dry air or of air mixed with aqueous vapour; let p', ϱ', τ' denote, as before, the pressure, the density, and the temperature, at the surface of the earth; and put p, ϱ, τ for the like quantities at the elevation z above the surface: the equations of equilibrium are these two, the radius of the earth being represented by a, viz.

$$p = \int \frac{-\,d\,z \,.\, \varrho}{\left(1 + \frac{z}{a}\right)^2},$$

$$\frac{p}{p'} = \frac{1+\beta\,\tau}{1+\beta\,\tau'} \cdot \frac{\varrho}{\varrho'}.$$

The second of these equations has already been noticed: the integral in the first being extended to the top of the atmosphere, is equal to the weight of the column of air above the initial height, every infinitesimal mass being urged by a gravitation which is equal to unit at the earth's surface, and decreases in the inverse proportion of the square of the distance from the earth's centre. By putting

$$\frac{1+\beta\tau}{1+\beta\tau'} = 1 - q, \quad \sigma = \frac{z}{1+\frac{z}{a}}, \quad \frac{\varrho}{\varrho'} = c^{-u},$$

the same two equations will be thus written, viz.

$$p = \varrho' \int - d\sigma\, c^{-u},$$

$$p = p'(1-q)\, c^{-u}.$$

The three quantities u, q, σ, are severally equal to zero at the earth's surface: and the two values of p will not be identical, unless the same three quantities can be expressed by functions of one variable, or, which is equivalent, unless two of them, as q and σ, are each functions of the remaining one u. Now q being a function of u, we shall have,

$$q = \frac{dq}{du} \cdot u + \frac{ddq}{du^2} \cdot \frac{u^2}{1.2} + \frac{d^3q}{du^3} \cdot \frac{u^3}{1.2.3} \text{ \&c.},$$

the differentials being valued when $u = 0$, that is, the particular values which they have at the earth's surface being taken. According to what was before shown, we have this other series for q, viz.

$$q = fu - (f - f') \cdot \frac{u^2}{1.2} + (f - 2f' + f'') \cdot \frac{u^3}{1.2.3} \text{ \&c.}:$$

and as the two series must be identical, it follows that the quantities f, f', f'', &c., will be known, if we can ascertain the particular values assumed at the surface of the earth by the differentials of q considered as varying with u, or with the density. Thus the coefficients in the formula (7.) are not hypothetical quantities, but such as have a real existence in nature, and which might be determined experimentally, if we had the means of observing the phenomena of the atmosphere with sufficient exactness, so as to be able to determine q when u is given. It is further to be observed, that the same formula is general for all atmospheres, whether the air be entirely dry, or mixed with aqueous vapour: for it has been investigated from equations common to all atmospheres in equilibrium, without any consideration of a particular state of the air.

By substituting the series for q in the equation

$$\frac{p}{p'} = (1-q)\, c^{-u},$$

we obtain,

$$\frac{p}{p'} = c^{-u} - f c^{-u} \mathrm{R}_1 - f' \frac{d \,.\, c^{-u} \mathrm{R}_3}{du} - f'' \,.\, \frac{d^2 \,.\, c^{-u} \mathrm{R}_5}{du^2} - \text{\&c.} \quad . \quad . \quad . \quad . \quad (8.)$$

Further, if this value of $\frac{p}{p'}$ be substituted in the equation

$$\frac{p}{p'} = \frac{\varrho'}{p'} \int - d\sigma \, c^{-u},$$

we shall find

$$\int - d\sigma \, c^{-u} = \frac{p'}{\varrho'} \cdot \left\{ c^{-u} - f \, . \, c^{-u} \, \mathrm{R}_1 - f' \frac{d \, . \, c^{-u} \, \mathrm{R}_3}{d u} - \&\text{c.} \right\}.$$

Now, let this expression be differentiated; then divided by c^{-u}; and, finally, integrated, attending to the nature of the functions concerned; and the following result will be obtained:

$$\sigma = \frac{z}{1 + \frac{z}{a}} = \frac{p'}{\varrho'} \left(u - f \cdot \frac{d \, . \, c^{-u} \, \mathrm{R}_2}{c^{-u} \, . \, d u} - f' \frac{d d \, . \, c^{-u} \, \mathrm{R}_4}{c^{-u} \, . \, d u_2} - \&\text{c.} \right) \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

The equations (7.), (8.), (9.) contain the theoretical explanation of the properties of the atmosphere. What is said may easily be proved by applying them to such phenomena as have been ascertained in a satisfactory manner. This application is besides necessary for determining the numerical values of the coefficients f, f', f'', &c., which enter into the expression of the refraction. For this purpose it is requisite to find the relations that subsist between the pressure, the temperature, and the height above the earth's surface, by combining the equations so as to exterminate u.

By performing the differentiations in the equation (9.), there will be obtained,

$$\sigma = \frac{p}{\rho} \left\{ u + f (\mathrm{R}_1 + \mathrm{R}_2) - f' (\mathrm{R}_2 + 2 \, \mathrm{R}_3 + \mathrm{R}_4) + \&\text{c.} : \right.$$

and, by expanding the functions,

$$\sigma = \frac{p'}{\rho'} \cdot \left\{ (1 + f) \, u - (2f - f') \cdot \frac{u^2}{2} + (2f - 3f' + f'') \frac{u^3}{1 \, . \, 2 \, . \, 3} - \&\text{c.} \right\}.$$

Now, by reverting the series for q, we get

$$u = \frac{q}{f} + \frac{f - f'}{2f} \cdot \frac{q^2}{f^2} + \frac{2f^2 - 4ff' + 3f'^2 - ff''}{6f^2} \cdot \frac{q^3}{f^3} + \&\text{c.};$$

and, by substituting this value of u, the following formula will be obtained:

$$\sigma = \frac{z}{1 + \frac{z}{a}} = \frac{p'}{\rho'} \cdot \left(\frac{1 + f}{f} \cdot q + \frac{f - f' - f^2}{2 f^3} \cdot q^2 + \&\text{c.} \right). \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A.})$$

This equation between the perpendicular elevation z, and the difference of temperature

$$q = \frac{\beta \, (\tau' - \tau)}{1 + \beta \, \tau'},$$

contains the law according to which the heat decreases as the height above the earth's surface increases.

Further, from the equation

$$\frac{p}{p'} = (1 - q)\, c^{-u},$$

we deduce

$$\log\left(\frac{p'}{p}\right) = u + \log\frac{1}{1-q} = u + q + \frac{q^2}{2} + \frac{q^3}{3} + \&c.;$$

and, by substituting the value of q,

$$\log\frac{p'}{p} = (1+f)\, u - \frac{f - f' - f^2}{2}\, u^2 + \frac{f - 2f' + f'' - 3f^2 + 3ff' + 2f^3}{6}\, u^3, \&c.$$

By means of this series and the value of σ in terms of u already found, it is easy to deduce

$$\sigma = \log\left(\frac{p'}{p}\right) \times \frac{p'}{\varrho'}\left(1 - \frac{f}{2}\cdot u + \frac{2f - 2f' + 3f^2 - 3ff' - f^3}{12(1+f)}\cdot u^2 - \&c.\right);$$

and, by substituting the value of u, we finally obtain

$$\sigma = \frac{z}{1 + \frac{z}{a}} = \log\frac{p'}{p} \times \frac{p'}{\varrho'}\cdot\left(1 - \frac{1}{2}\, q - \frac{f + f^3 - f'}{12 f^2 (1+f)}\cdot q^2, \&c.\right). \quad . \quad . \quad (\mathrm{B.})$$

This formula determines a perpendicular ascent z, when the difference of the pressures, and of the temperatures, at its upper and lower extremities, have been found.

The formulas that have been investigated are true in an atmosphere of air mixed with aqueous vapour, as well as in one of perfectly dry air; but in applying them, perspicuity requires that the two cases be separately considered.

Atmosphere of dry air.

In applying the formula (A.) to the experimental ascents that have been made in the atmosphere, σ may be accounted equal to z, the height ascended: for $\frac{z}{a}$, which is a minute fraction at the top of the atmosphere, is insensible in small elevations. Further, in such experiments, the depression of the thermometer, or the difference of the temperature at the upper and lower extremities of the ascent, is only a moderate number of degrees; and as β is a very small fraction, the value of q in the formula

$$q = \frac{\beta(\tau' - \tau)}{1 + \beta\tau'},$$

will be so inconsiderable, that its powers may be neglected. Attending to what is said, the formula (A.), even in those cases where the ascents are most considerable, may take this very simple form without much error, or rather with all the accuracy warranted by the nature of such experiments, viz.

$$z = \frac{p'}{\varrho'}\cdot\frac{1+f}{f}\cdot\frac{\beta(\tau' - \tau)}{1 + \beta\tau'};$$

or, by making $\mathrm{D} = \varrho'(1 + \beta\tau')$,

$$z = \frac{1+f}{f}\cdot(\tau' - \tau)\,.\,\beta\cdot\frac{p'}{\mathrm{D}}.$$

Now it is obvious that D is the density of the air at the earth's surface, reduced to zero of the thermometer; and hence we learn that $\frac{p'}{\mathrm{D}}$ is independent on the magnitude of p', and has the same value in all atmospheres of dry air; for, D being the density of the air produced by the pressure p' at the fixed temperature zero of the thermometer, it will vary proportionally to p'.

The value of the constant quantity $\frac{p'}{\mathrm{D}}$ is next to be found. It has been ascertained, by very careful experiments, that the density of mercury is to the density of dry air as 10462 to 1, the temperature being 0° centigrade, or 32° of FAHRENHEIT's scale, and the barometric pressure $0^{\mathrm{m}}\cdot76$, or 29·9218 English inches. The temperature remaining at 32° FAHRHENHEIT, if the barometric pressure be changed to p', the density of mercury will be to the density of dry air, at the temperature 32° FAHRHENHEIT and under the pressure p', as $10462 \times \frac{29\cdot9218}{p'}$ to 1; wherefore, as D stands for the density of dry air in the circumstances mentioned, its value estimated in parts of the density of mercury, will be thus expressed:

$$\mathrm{D} = \frac{1}{10462} \times \frac{p'}{29\cdot9218}:$$

hence

$$\frac{p'}{\mathrm{D}} = 10462 \times 29\cdot9218;$$

and, by reducing the inches to fathoms,

$$\frac{p'}{\mathrm{D}} = \mathrm{L} = 4347\cdot8 \text{ fath.}$$

This quantity being found, we deduce from the foregoing formula for z,

$$\frac{1+f}{f} = \frac{1}{\beta \mathrm{L}} \times \frac{z}{\tau' - \tau}.$$

A single experiment in which z and $\tau' - \tau$ were ascertained, should be sufficient for determining $\frac{1+f}{f}$ and f: but it is well known that great irregularity prevails in the rate at which the heat decreases in the atmosphere, more especially when the elevations are small. This is owing chiefly to the thermometer, which is often affected by local and temporary causes. When we reflect that a considerable variation in the height is required to produce a small change of the thermometer, even the errors unavoidable in the use of that instrument must produce notable discrepancies in the rate, when the whole observed difference of temperature is only a few degrees. It thus appears that the quantity sought cannot be determined with tolerable exactness, except by taking a mean of the results obtained from many experiments. In this view, the average estimations of the decrease of heat in the atmosphere, which have been inferred from their own researches by philosophers on whose judgement and accuracy dependence can be had, becomes very valuable. Professor PLAYFAIR, in his

Outlines, states that the decrease of heat is nearly uniform for the greatest heights we can reach; and that it may be taken on an average as equal to 1° of FAHRENHEIT'S thermometer for 270 feet, or 45 fathoms, of perpendicular ascent. The same rate has the authority of Professor LESLIE, to whom Meteorology is so much indebted. If we make $z = 45$ fathoms, $\tau' - \tau = 1°$, $\beta = \frac{1}{480}$, we shall obtain

$$\frac{1+f}{f} = \frac{45}{9 \cdot 05} = 5, f = \frac{1}{4},$$

which are the numbers assumed in the paper of 1823.

According to Dr. DALTON, another eminent philosopher who has studied meteorology very successfully, and made many experiments with great care, the average ascent for depressing FAHRENHEIT'S thermometer 1° is 300 feet, or 50 fathoms: this gives

$$\frac{1+f}{f} = \frac{50}{9 \cdot 05} = 5 \cdot 5.$$

RAMOND, in his Treatise on the Barometrical Formula, has recorded the heights for depressing the centigrade thermometer 1°, in 42 different experiments. Setting aside four of this great number on account of their excessive irregularity, he states the mean of the remaining 38 at $164^{\mathrm{m}} \cdot 7$. A good average may be expected from so many experiments, made by observers of the greatest eminence, in different quarters of the world, in every variety of height and temperature. Now 4347·8 fathoms $= 7951^{\mathrm{m}}$; $\beta = \frac{3}{800}$; $z = 164 \cdot 7$; consequently

$$\frac{1+f}{f} = \frac{800 \times 164 \cdot 7}{7951 \times 3} = 5 \cdot 5$$

It would be a great omission in this research to leave out the celebrated ascent of GAY LUSSAC in a balloon. Acccording to LAPLACE, the whole height ascended, or z, is 6980^{m}, the depression of the thermometer, or $\tau' - \tau$, being $40^{\circ} \cdot 25$ centigrade: hence

$$\frac{1+f}{f} = \frac{800}{7951 \times 3} \times \frac{6980}{40 \cdot 25} = 5 \cdot 8.$$

It is to be observed that, although experience and theory both concur in proving that z and $\tau' - \tau$ increase together in the same proportion to considerable elevations in the atmosphere, yet, at very great elevations, there is no doubt that z increases in a greater ratio than $\tau' - \tau$: so that when very great heights are used for computing $\frac{1+f}{f}$, the resulting value will be greater than the true quantity. What is said accounts sufficiently for the excess of $\frac{1+f}{f}$ deduced from GAY LUSSAC'S ascent, above the other values found from moderate elevations. Without further research we may adopt the following determinations as near approximations derived from a multitude of experiments,

$$\frac{1+f}{f} = 5 \cdot 5 ; f = \frac{2}{9}.$$

The difference of these numbers from those used in the paper of 1823, produces an increase in the refractions, amounting to 19″ at the horizon, and to 2″ at 2° of altitude.

The irregular manner in which the heat decreases in such experiments as have been used for finding f, evidently makes them altogether unfit for determining the next coefficient f'. One remark respecting this quantity deserves to be noticed. By expanding the formula (4.), we obtain,

$$\frac{1+\beta\tau}{1+\beta\tau'} = 1 - fu + f\cdot\frac{u^2}{1.2} - f\cdot\frac{u^3}{1.2.3} + \&c.:$$

the exact value is

$$\frac{1+\beta\tau}{1+\beta\tau'} = 1 - fu + (f - f')\cdot\frac{u^2}{1.2} - \&c.:$$

now, as these values continue very nearly equal to considerable elevations, the first terms of the two series must nearly coincide: which requires that f' shall be only a small part of f.

We have next to attend to the formula (B.). As q is only a small fraction in all the elevations that have been reached in the atmosphere, its square and other powers may be neglected: so that,

$$\sigma = \frac{z}{1+\frac{z}{a}} = \frac{p'}{\rho'}\left(1 - \frac{q}{2}\right)\log\left(\frac{p'}{p}\right):$$

and, because

$$1 - \frac{q}{2} = \frac{1+\beta\left(\frac{\tau'+\tau}{2}\right)}{1+\beta\tau'}; \quad \frac{p'}{\rho'(1+\beta\tau')} = \mathrm{L}:$$

$$z = \left(1+\frac{z}{a}\right).\mathrm{L}\left(1+\beta\left(\frac{\tau'+\tau}{2}\right)\right).\log\left(\frac{p'}{p}\right).$$

Now this is nothing more than the usual barometric formula for measuring heights, as it is found in the writings of LAPLACE or POISSON. It supposes that unit represents the force of gravity at the earth's surface; and if the variable intensity of that force in different latitudes must be taken into account, nothing more is requisite than to multiply by the proper factor. When this is done the foregoing expression will be identical with the usual formula, all its minutest corrections included. But there is this difference between the two cases, that the usual formula is investigated on the arbitrary supposition that the temperature is constant at all the points of an elevation, and equal to the mean of the temperatures at the two extremities; whereas the other expression is strictly deduced from the general properties of an atmosphere in equilibrium. The exact theoretical formula has been made to coincide with the approximate one, by dismissing all the terms that cannot be estimated in the present state of our knowledge of the phenomena of the atmosphere.

All the properties of the atmosphere that have been ascertained with any degree of certainty, have been made known to us by the application of the barometric formula:

it would therefore be superfluous to attempt, by the consideration of particular experiments, any further elucidation of a theory which is, in a manner, identical with observation, as far as our knowledge extends.

Atmosphere of air mixed with aqueous vapour.

Continuing to represent the pressure and temperature at the earth's surface by p' and τ', and the like quantities at the height z by p and τ, the symbols (ϱ'), (ϱ) may be used to denote the respective densities in the case of air mixed with aqueous vapour. When the pressure and density vary, all the gases, and mixtures of gases and vapours, are found to follow the same laws of dilatation and compression: and hence the same equations that express the equilibrium of an atmosphere of dry air, will hold equally in one of moist air. In the present case these equations will therefore be,

$$p = \int \frac{-dz.(\rho)}{\left(1+\frac{z}{a}\right)^2},$$

$$\frac{p}{p'} = \frac{1+\beta\tau}{1+\beta\tau'} \cdot \frac{(\rho)}{(\rho')}\text{*}:$$

and if we put

$$\sigma = \frac{z}{1+\frac{z}{a}}, \quad \frac{1+\beta\tau}{1+\beta\tau'} = 1-q, \quad \frac{(\rho)}{(\rho')} = c^{-u},$$

the same equations will be thus written,

$$p = (\varrho')\int - d\sigma\, c^{-u},$$

$$p = p'(1-q)\,c^{-u}.$$

The three quantities σ, q, u are severally equal to zero at the surface of the earth: so that, by the same procedure as before, we shall obtain these formulas,

$$q = fu - (f-f')\frac{u^2}{2} + \&c.$$

$$\sigma = \frac{p'}{(\rho')} \cdot \left\{ u - f.\frac{d.c^{-u}R_2}{c^{-u}du} - f'.\frac{dd.c^{-u}R_2}{c^{-u}du^2} - \&c.\right.$$

But it is to be observed that, in these expressions, the coefficients f, f', &c., are not exactly the same as in an atmosphere of dry air: for the quantities mentioned, although they have determinate values in the same quiescent atmosphere, depend upon the manner in which the temperature q, or the height z, varies relatively to the density, or to u.

If we suppose that the height z is not very great, so that the powers of q may be

* This equation is equivalent to the one in p. 18 of M. BIOT's dissertation, on which that author lays so much stress.

neglected, we shall obtain from the foregoing equations,

$$z = \frac{p'}{(\rho')} \cdot \frac{1+f}{f} \cdot q:$$

and hence

$$\frac{1+f}{f} = \frac{1+\beta\tau'}{\beta} \cdot \frac{(\rho')}{p'} \cdot \frac{z}{\tau'-\tau}.$$

In order to ascertain how far this value is different from the like value in the case of dry air, we must resolve the complex density (ϱ') into its elements. The hygrometer will discover the tension of the vapour at the earth's surface; and if φ' denote this tension in inches of mercury, and ϱ' be the density of dry air under the pressure p' and at the temperature τ', the following equation is proved in all the late treatises on Natural Philosophy,

$$(\varrho') = \varrho'\left(1 - \frac{3}{8} \cdot \frac{\varphi'}{p'}\right):$$

by means of which we obtain

$$\frac{1+f}{f} = \frac{1 - \frac{3}{8} \cdot \frac{\Phi'}{p'}}{\beta \mathrm{L}} \times \frac{z}{\tau'-\tau},$$

$$\mathrm{L} = \frac{p'}{\rho'(1+\beta\tau')}.$$

Now the small additional factor in the value of $\frac{1+f}{f}$ is not taken into account in the measurement of heights by the barometer, no distinction being usually made between dry air and moist air. In order to form some estimate of its effect, we may instance the mean atmosphere of our climate, the temperature of which is 50° Fahrhenheit; the greatest possible tension of vapour in such an atmosphere is ·36 of an inch of mercury; at a medium, if we make $\varphi' = \cdot 18$, and $p' = 30$ inches, we shall have,

$$1 - \frac{3}{8} \cdot \frac{\Phi'}{p'} = 1 - \frac{1}{444}.$$

It thus appears that in our climate, when the mean portion of aqueous vapour is mixed with the air, the value of $\frac{1+f}{f}$ is less than it would be if the air were perfectly dry by its $\frac{1}{444}$th part, a quantity too minute to be perceptible in most experiments. A small part only of the refractions depend upon f, about a twelfth part of the whole at the horizon; so that, neglecting the minute variations which f undergoes by the greater or less portions of aqueous vapour mixed with the air, the effect of which on the refractions is insensible, we may assume that it has the same value in all atmospheres. The same thing applies with greater force to the other coefficients f', f'', &c., which having themselves hardly any influence on the refractions, their minute changes in different atmospheres may be wholly disregarded.

If we substitute for (ϱ') its equivalent $\varrho'\left(1 - \frac{3}{8} \cdot \frac{\varphi'}{p'}\right)$ in the foregoing value of σ,

we shall obtain the following equation, which is sufficient for the problem of the refractions in an atmosphere of moist air:

$$\sigma = \frac{1}{1 - \frac{3}{8} \cdot \frac{\phi'}{p'}} \cdot \frac{p'}{\rho'} \cdot \left\{ u - f \cdot \frac{d \, . \, c^{-u} \mathrm{R}_2}{c^{-u} d u} - f' \cdot \frac{d d \, . \, c^{-u} \mathrm{R}_4}{c^{-u} d u^2} - \&c. \right\} . \quad . \quad . \quad (10.)$$

In which expression the coefficients f, f', &c., may be considered the same in all atmospheres, the quantity u varying from zero at the earth's surface to be infinitely great at the top of the atmosphere.

8. In the foregoing analysis, every formula has been strictly deduced from the equations of equilibrium: no quantities have been introduced except such as really exist in nature, and might be determined experimentally, if we had the means of exploring the phenomena of the atmosphere with the requisite accuracy. It may not be improper to notice here an obvious consequence of the equation

$$p = \varrho' \int - d \sigma c^{-u},$$

which holds in an atmosphere of dry air; namely, that the integral

$$\int - d \sigma c^{-u},$$

being extended from the surface of the earth to the top of the atmosphere, is the analytical expression of $\frac{p'}{\varrho'}$, or of the height of the homogeneous atmosphere, that is, of a column of air equiponderant to the whole atmosphere, and every part of which has the same density and the same weight which it would have at the surface of the earth. This height varies only with the temperature, and is thus determined:

$$\frac{p'}{\rho'} = \frac{p'}{\rho' (1 + \beta \tau')} \cdot (1 + \beta \tau') = \frac{p'}{\mathrm{D}} (1 + \beta \tau') = \mathrm{L} (1 + \beta \tau').$$

In like manner, in an atmosphere of air mixed with aqueous vapour, the same integral is equal to $\frac{p'}{(\rho')}$: and we have

$$\frac{p'}{(\rho')} = \frac{p'}{\rho'} \cdot \frac{1}{1 - \frac{3}{8} \cdot \frac{\phi'}{p'}} = \frac{\mathrm{L} (1 + \beta \tau')}{1 - \frac{3}{8} \cdot \frac{\phi'}{p'}}.$$

Thus the analytical theory agrees in every respect with the real properties of the atmosphere, as far as these have been ascertained; and we now proceed to show that the same theory represents the astronomical refractions with a fidelity that can be deemed imperfect only in so far as the constants f, f', &c., which can only be determined by experiment, are liable to the charge of inaccuracy.

9. The apparent zenith-distance of a star being represented by θ, and the refraction by $\delta \theta$, the following formulas have already been obtained (§ 2. equations (2.) and (3.)).

$$d \, . \, \delta \theta = \frac{d y}{\sqrt{r^2 - y^2}},$$

$$y = a \sin \theta \times \sqrt{\frac{1 + 2 \phi (\rho')}{1 + 2 \phi (\rho)}},$$

the quantity $\delta \theta$ being supposed to increase from the surface of the earth to the top of the atmosphere. For the sake of perspicuity, we shall, in the first place, confine our attention to an atmosphere of dry air, in which case it is known by experiment that the refractive power $\varphi (\varrho)$ is proportional to the density ϱ; so that

$$\varphi (\varrho) = \mathrm{K} \times \varrho,$$

K being a constant. Adverting to the mode of expression before used, we have

$$\varrho = \varrho' c^{-u};$$

and hence

$$\varphi (\varrho) = \mathrm{K} \times \varrho = \mathrm{K} \varrho' . c^{-u},$$

$$y = a \sin \theta \times \sqrt{\frac{1 + 2 \mathrm{K} \rho'}{1 + 2 \mathrm{K} \rho' c^{-u}}}:$$

and by introducing new symbols in order to abridge expressions,

$$\alpha = \frac{\mathrm{K} \rho'}{1 + 2 \mathrm{K} \rho'},$$

$$\omega = 1 - c^{-u},$$

$$y = \frac{a \sin \theta}{\sqrt{1 - 2 \alpha \omega}}.$$

Let this value of y be substituted in the differential of the refraction; then

$$r^2 = (a + z)^2 = a^2 \left(1 + \frac{z}{a}\right)^2 = \frac{a^2}{\left(1 - \frac{\sigma}{a}\right)^2},$$

$$d . \delta \theta = \sin \theta \times \frac{\alpha}{1 - 2 \alpha \omega} \times \frac{d \omega}{\sqrt{\frac{1 - 2 \alpha \omega}{\left(1 - \frac{\sigma}{a}\right)^2} - \sin^2 \theta}}.$$

In further transforming this expression, it is to be observed that α is a very small fraction less than $\cdot 0003$; and if the atmosphere extend fifty miles above the earth's surface, $\frac{z}{a}$ or $\frac{\sigma}{a}$ when greatest will not exceed $\cdot 012$. If we now put

$$\frac{\sigma}{a} = \frac{s}{a} + \alpha \omega,$$

we shall have

$$\frac{1 - 2 \alpha \omega}{\left(1 - \frac{\sigma}{a}\right)^2} = \frac{(1 - \alpha \omega)^2 - \alpha^2 \omega^2}{\left(1 - \alpha \omega - \frac{s}{a}\right)^2} = 1 + 2 \frac{s}{a} + 3 \frac{s^2}{a^2},$$

the quantities rejected being plainly of no account relatively to those retained. Further, because ω is always less than 1, $\frac{\alpha}{1 - 2 \alpha \omega}$ is contained between α and $\alpha (1 + 2 \alpha)$; and

it may be taken equal to α, or to the mean value $\alpha(1+\alpha)$. Thus we have

$$d.\delta\theta = \sin\theta \times \frac{\alpha(1+\alpha)\, d u\, c^{-u}}{\sqrt{\cos^2\theta + 2\frac{s}{a} + 3\frac{s^2}{a^2}}}.$$

Again, the formula (9.) gives

$$\sigma = s + a.\alpha\omega = \frac{p'}{\rho'}\cdot\left\{u - f'.\frac{d.c^{-u}R_2}{c^{-u}du} - f'.\frac{dd.c^{-u}R_4}{c^{-u}.du^2} - \&c.\right\}.$$

Now,

$$\frac{p'}{\rho'} = \frac{p'}{\rho'(1+\beta\tau')}\cdot(1+\beta\tau') = L(1+\beta\tau'):$$

and if we make

$$s.\frac{\varrho'}{p'} = \frac{s}{L(1+\beta\tau')} = x,$$

$$\frac{p'}{\rho'}\cdot\frac{1}{a} = \frac{L(1+\beta\tau')}{a} = i,$$

$$\frac{a.\rho'.\alpha}{p'} = \frac{\alpha}{i} = \lambda,$$

we shall have

$$\frac{s}{a} = i x$$

$$x = u - \lambda(1-c^{-u}) - f.\frac{d.c^{-u}R_2}{c^{-u}du} - f'\frac{dd.c^{-u}R_4}{c^{-u}du^2} - \&c.$$

Let $\Psi(u)$ stand for all the terms in this value of x except the first, so that

$$x = u - \Psi(u):$$

from this we deduce by LAGRANGE's theorem,

$$c^{-u} = c^{-x} - c^{-x}\Psi(x) - \frac{1}{2}\cdot\frac{d.c^{-x}\Psi^2(x)}{dx} - \&c.:$$

consequently,

$$du\,c^{-u} = dx\,c^{-x} + \frac{d.c^{-x}\Psi(x)}{dx}dx + \frac{1}{2}\cdot\frac{dd.c^{-x}\Psi^2(x)}{dx^2}dx + \&c.$$

By means of the values that have been found, the differential of the refraction can be expressed in terms of one variable x. In making the substitutions, the smallest term of the radical quantity is to be neglected in all the terms of $d u\, c^{-u}$, except the first and greatest; and the denominator of that term is to be expanded. Thus we obtain

$$d.\delta\theta = \sin\theta.\alpha(1+\alpha).\left\{\int\frac{dx}{\sqrt{\cos^2\theta + 2ix}}\cdot\left(c^{-x} + \frac{d.c^{-x}\Psi(x)}{dx}\right)\right.$$

$$+\frac{1}{2}\int\frac{dx}{\sqrt{\cos^2\theta+2ix}}\cdot\frac{dd.c^{-x}\Psi^2(x)}{dx}$$

$$-\frac{3}{2}\int\frac{dx.c^{-x}.i^2x^2}{(\cos^2\theta+2ix)^{\frac{3}{2}}}.$$

In order to estimate the relative magnitude of the several parts of this formula we

must find the numerical values of the quantities α and i. If η stands for the refraction at 45° of altitude, determined very exactly from many astronomical observations, we shall have

$$\alpha = \eta\,(1 - 2\,i + 2\,\eta),$$

as will readily appear from the formula according to CASSINI's method given in § 1. MM. BIOT and ARAGO have ascertained the value of α with great exactness in a different way, by means of experiments on the gases with the prism. In some of the best attempts to determine α, the refractions at 45° of altitude, being reduced to the barometer 29·6 and to the temperature 50° FAHR., are as follows:

Dr. BRINKLEY	57″·42
DE LAMBRE	57·58
BESSEL, Tab. Reg	57·55
Experiments of MM. BIOT and ARAGO	57·65
Mean	57·55

It appears that BESSEL's determination has the best claim to be preferred: but as it differs very little from DE LAMBRE's result, which is adopted in the paper of 1823, the same value will be retained in the calculations which follow. According to DE LAMBRE, the value of α is 60″·616* at the temperature 0° centigrade, and the barometric pressure $0^{\mathrm{m}}\cdot 76$: wherefore, when the temperature is 50° FAHR. and the pressure 30 inches ($= 0^{\mathrm{m}}\cdot 762$), we shall have

$$\alpha = 60\cdot 616 \times \frac{762}{760 \times 1\cdot 0018} \times \frac{1}{1 + \frac{18}{480}} = 58''\cdot 47:$$

and in parts of the radius,

$$\alpha = \cdot 0002835.$$

It has been found that L = 4347·8 fathoms at 0° centigrade or 32° FAHR.: wherefore, if we make a = mean radius of the earth = 3481280 fathoms, we shall have at the temperature of our climate, or 50° FAHR.,

$$i = \frac{\mathrm{L}\,(1 + \beta\,\tau')}{a} = \frac{4347\cdot 8\left(1 + \frac{18}{483}\right)}{3481280} = \cdot 0012958;$$

and hence

$$\lambda = \frac{\alpha}{i} = \cdot 21878.$$

We can now inquire into the values of the last two terms of the foregoing formula for the refraction, both of which are very small. With respect to the first of them, we have,

$$\Psi\,(x) = \lambda\,(1 - c^{-x}) + f \cdot \frac{d\,.\,c^{-x}\,\mathrm{R}_2}{c^{-x}\,d\,x} + f' \cdot \frac{d\,d\,.\,c^{-x}\,\mathrm{R}_4}{c^{-x}\,d\,x^2} - \&c.:$$

* Tableaux Chronomiques, publiées par le Bureau des Longitudes de France.

and, by performing the differential operations,

$$\Psi(x) = \lambda(1 - c^{-x}) + f(R_1 + R_2) + f'(R_2 + 3 R_3 + R_4);$$

and, by substituting the values of the functions,

$$h = 2f - \lambda = \cdot 22566$$

$$\Psi(x) = -h(1 - c^{-x}) + fx + 4f'\left(1 - x + \frac{3x^2}{8} - \frac{x^3}{24} - c^{-x}\right).$$

It might not be very objectionable to neglect the term multiplied by f', for the same reasons that the terms which follow it are neglected, that is, both on account of the nature of the functions and because the coefficients are small: but, in order to leave no room for scruples respecting accuracy, the square of the entire expression set down, may be thus represented:

$$\Psi^2(x) = G - 8hf'.G' + 8ff'.G'' + 16f'^2.G'''.$$

The integral in the term under consideration is greatest when the radical quantity in the denominator is least, that is, when $\cos\theta = 0$: and if the integration be performed betwen the limits $x = 0$, $x = \infty$, we shall obtain a result greater than if the integral were extended only to the top of the atmosphere. Now we have,

$$G = h^2(1 - 2c^{-x} + c^{-2x}) + 2hf.xc^{-x} - 2hf.x + f^2.x^2:$$

and, by operating on the terms separately, the part of the integral depending on G, will be as follows:

$$\int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{dd.c^{-x}G}{dx^2} =$$

$$\frac{\sqrt{\pi}}{\sqrt{2i}} \times \left(h^2(1 - 4\sqrt{2} + 3\sqrt{3}) - 3hf(\sqrt{2} - 1) + \frac{3}{4}f^2\right) = \frac{\sqrt{\pi}}{\sqrt{2i}} \times \cdot 00216.$$

The other parts depending on G′, G″, G‴ are complicated; but they are troublesome more on account of the number of terms they contain than from any difficulty in the integrations. The following results have been obtained:

$$8hf' \times \int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{dd.c^{-x}G'}{dx^2} = -f' \times \frac{\sqrt{\pi}}{\sqrt{2i}} \times \cdot 01759,$$

$$8ff' \times \int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{dd.c^{-x}G''}{dx^2} = -f' \times \frac{\sqrt{\pi}}{\sqrt{2i}} \times \cdot 02043,$$

$$16f'^2 \int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{dd.c^{-x}G'''}{dx^2} = +f'^2 \times \frac{\sqrt{\pi}}{\sqrt{2i}} \times \cdot 00855.$$

Collecting all the parts, the term sought is found, viz.

$$\frac{\alpha(+\alpha)}{2} \cdot \int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{dd.c^{-x}\Psi^2(x)}{dx^2} =$$

$$\frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}} \times \left(\cdot 00108 - f' \times \cdot 00142 + f'^2 \times \cdot 00427\right).$$

To this must be added the other term, which, being integrated in the same circumstances, gives,

$$-\frac{3}{2}\cdot\int_0^{\infty}\frac{d\,x\,c^{-x}}{\sqrt{2\,i\,x}}\times\frac{i\,x}{2}=-\frac{3}{8}\cdot\frac{\sqrt{\pi}}{\sqrt{2\,i}}=-\frac{\sqrt{\pi}}{\sqrt{2\,i}}\times\cdot 00049.$$

It thus appears that the two small terms of the expression of the refraction are, together, equal to

$$\frac{\alpha\,(1+\alpha)\,\sqrt{\pi}}{\sqrt{2\,i}}\cdot\left(\cdot 00059-f'\times\cdot 00142+f'^2\times\cdot 00427\right):$$

and as

$$\frac{\alpha\,(1+\alpha)\,\sqrt{\pi}}{\sqrt{2\,i}}=2036''\cdot 5,$$

the greatest amount of both is about 1''.

The whole refraction will therefore be thus expressed:

$$d\,.\,\delta\,\theta=\sin\theta\times\alpha\,(1+\alpha)\int\frac{d\,x}{\sqrt{\cos^2\theta+2\,i\,x}}\cdot\left(c^{-x}+\frac{d\,.\,c^{-x}\,\Psi\,(x)}{d\,x}\right),$$

with the assurance that the error cannot exceed 1''. If we substitute what $\Psi\,(x)$ stands for, we shall have

$$d\,.\,\delta\,\theta=\sin\theta\times\alpha\,(1+\alpha)\times\int\frac{d\,x}{\sqrt{\cos^2\theta+2\,i\,x}}\times$$

$$\left(c^{-x}+\lambda\cdot\frac{d\,.\,(c^{-x}-c^{-2x})}{d\,x}+f\cdot\frac{d\,d\,.\,c^{-x}\,\mathrm{R}_2}{d\,x^2}+f'\cdot\frac{d^3\,.\,c^{-x}\,\mathrm{R}_4}{d\,x^3}+\&c.\right).$$

This expression being regular, it may be continued to any number of terms, and it has the advantage of being linear with respect to the coefficients. Adverting to what x stands for, it will appear that $\mathrm{L}\times x$ is nearly equal to s, or to z, that is, to the elevation in the atmosphere; so that, if we suppose the greatest height of the atmosphere is $10\times\mathrm{L}$, or about fifty miles, the greatest value of x will be 10; and all the integrals in the foregoing expression must be taken between the limits zero and 10. But the quantity c^{-x} is so small when x has increased to 8 or 10, that the results are not sensibly different whether the integrals be extended to those limits or be continued to infinity. By substituting the values of the functions, the expression of $\delta\,\theta$ will take this form:

$$\delta\,\theta=\sin\theta\times\alpha\,(1+\alpha)\times\left\{\int\frac{d\,x\,c^{-x}}{\sqrt{\cos^2\theta+2\,i\,x}}+\lambda\int\frac{d\,x\,(2\,c^{-2x}-c^{-x})}{\sqrt{\cos^2\theta+2\,i\,x}}\right.$$

$$-f\int\frac{d\,x}{\sqrt{\cos^2\theta+2\,i\,x}}\cdot\left(4\,c^{-2x}-3\,c^{-x}+x\,c^{-x}\right)$$

$$+f'\int\frac{d\,x}{\sqrt{\cos^2\theta+2\,i\,x}}\cdot\left(8\,c^{-2x}-8\,c^{-x}+7\,x\,c^{-x}-2\,x^2\,c^{-x}+\frac{x^3\,c^{-x}}{6}\right)$$

$$-f''\int\frac{d\,x}{\sqrt{\cos^2\theta+2\,i\,x}}\cdot\left(16\,c^{-2x}-16\,c^{-x}+16\,x\,c^{-x}-\frac{15}{2}\,x^2\,c^{-x}\right.$$

$$\left.+\frac{11}{6}\,x\,c^{-x}-\frac{5}{24}\,x^4\,c^{-x}+\frac{x^5\,c^{-x}}{120}\right).$$

In order to illustrate the rapidity with which the terms decrease, it may be proper to find the limit of $\delta\,\theta$, by making $\cos^2\theta = 0$, and integrating between the limits $x = 0$, $x = \infty$; which limit is not sensibly different from the refraction at the horizon. Now it will be found that, in the circumstances mentioned,

$$\begin{aligned}\delta\,\theta = \frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2\,i}} \times \Big\{ 1 &+ \lambda\left(\sqrt{2} - 1\right) \\ &- f\left(2\sqrt{2} - \tfrac{5}{2}\right) \\ &+ f'\left(4\sqrt{2} - \tfrac{91}{16}\right) \\ &- f''\left(8\sqrt{2} - \tfrac{2895}{256}\right) \\ &- \&c.:\end{aligned}$$

or, in seconds,

$$\delta\,\theta = 2072''{\cdot}46 - f' \times 62{\cdot}4 - f'' \times 10''{\cdot}2 - \&c.$$

From this calculation it appears that the term multiplied by f'' and all the subsequent terms are too small to be sensible; and as f' is much less than f, even the term multiplied by f' can hardly exceed a few seconds at low altitudes. There is great probability that the horizontal refraction is very near $34'\,30''$, and does not exceed this quantity.

To prepare the foregoing expression of $\delta\,\theta$ for integration, put

$$m = 10,\ \frac{\sqrt{2\,i\,m}}{\cos\theta} = \tan\varphi,\ \ e = \tan\frac{\varphi}{2};$$

then

$$\cos^2\theta = \frac{(1-e^2)^2}{4\,e^2} \times 2\,i\,m,$$

$$\sqrt{\cos^2\theta + 2\,i\,m} = \frac{\sqrt{5\,i}}{e}\cdot\sqrt{(1-e^2)^2 + 4\,e^2\cdot\frac{x}{m}} = \frac{\sqrt{5\,i}}{e}\cdot\Delta:$$

and we shall have

$$\left.\begin{aligned}\delta\,\theta = \sin\theta \times \frac{\alpha(1+\alpha)}{\sqrt{5\,i}} \times \Big\{&\int_0^m \frac{e\,d\,x}{\Delta}\cdot c^{-x} \\ &+ \lambda\int_0^m \frac{e\,d\,x}{\Delta}\cdot\left(2\,c^{-2x} - c^{-x}\right) \\ &- f\int_0^m \frac{e\,d\,x}{\Delta}\cdot\left(4\,c^{-2x} - 3\,c^{-x} + x\,c^{-x}\right) \\ &+ f'\int_0^m \frac{e\,d\,x}{\Delta}\cdot\left(8\,c^{-2x} - 8\,c^{-x} + 7\,x\,c^{-x} - 2\,x^2\,c^{-x} + \frac{x^3}{6}\,c^{-x}\right).\end{aligned}\right\} \quad (\text{C.})$$

For the sake of abridging, the several integrals in succession may be represented by Q_0, Q_1, Q_2, Q_3; so that the value of $\delta\,\theta$ will be thus written:

$$\delta\,\theta \sin\theta \times \frac{\alpha(1+\alpha)}{\sqrt{5\,i}}\cdot(Q_0 + \lambda\,Q_1 - f\,Q_2 + f'\,Q_3).$$

10. The equation (C.) supposes that the atmosphere consists entirely of dry air: we have next to consider what modification must be made when it contains a portion of aqueous vapour.

In the first place, when p' and τ', the pressure and temperature at the surface of the earth, are given, as they are in the mean atmosphere which produces the refractions, the quantity α, or the refractive power of the air, is not liable to be altered by any possible mixture of aqueous vapour. For if an addition of vapour to dry air diminish the refractive power by making the density less, the greater action of the vapour upon light is found almost exactly to compensate the defect. LAPLACE first made this observation; which has been confirmed by MM. BIOT and ARAGO, who have established by experiments, that the refractive power of air, whether dry or mixed with vapour, is the same, when the pressure and temperature are the same. It thus appears that, as far as the quantity α, or the refractive power of the air at the earth's surface, is concerned, the astronomical refractions are independent on the hygrometric condition of the atmosphere.

But a mixture of vapour may produce changes in the expression of the refraction, by altering the coefficients or the integrals. Now, if we attend to the formulas that have been found for an atmosphere of moist air, and in the equation (10.) make the same substitution as in the case of dry air, viz.

$$\frac{\sigma}{a} = \frac{s}{a} + \alpha\,\omega,$$

we shall obtain

$$s \cdot \frac{(\rho')}{p'} = \frac{s\left(1 - \frac{3}{8}\frac{\varphi'}{p'}\right)}{\mathrm{L}(1+\beta\tau')} = x,$$

$$\frac{p'}{(\rho')}\cdot\frac{1}{a} = \frac{\mathrm{L}(1+\beta\tau')}{a\left(1-\frac{3}{8}\cdot\frac{\varphi'}{p'}\right)} = i,$$

$$\alpha \cdot \frac{a\,(\rho')}{p'} = \frac{\alpha}{i} = \frac{a\left(1-\frac{3}{8}\cdot\frac{\varphi'}{p'}\right)}{\mathrm{L}(1+\beta\tau'} = \lambda:$$

and further, it will appear that the same relation subsists between x and u in the atmosphere of moist air as between the quantities represented by the same letters in the atmosphere of dry air. The same procedure will therefore lead, in both cases, to the same integrals extending between the same limits. The only difference lies in the values of λ and i, which in the case of moist air acquire, as a multiplier or divisor, the small factor $\left(1 - \frac{3}{8}\cdot\frac{\phi'}{p'}\right)$ depending on the tension of the vapour at the earth's surface. If the hygrometer afforded an easy practical method of ascertaining the tension of the vapour, the minute variations of the refractions, arising from moisture in the atmosphere, might be corrected by the method usually employed for compen-

sating the small changes which a difference of temperature causes in the mean constants.

Experience confirms what has been said; for all the astronomers who have attended to aqueous vapour in the atmosphere, agree in admitting that it either has no influence, or but a very small and imperceptible effect, to alter the refractions. On this head it will be sufficient to cite the authority of M. Biot*, who seems carefully to have studied this point, on which he expresses himself very strongly. The very exact coincidence of the theoretical with the observed refractions as far as 88° or $88\frac{1}{2}°$ from the zenith, concurs to prove that the variable quantity of vapour in the air has little influence so long as it retains the gaseous form; but at lower altitudes, when the rays of light become almost parallel to the horizon, it is very probable that particular and local causes may come into play.

11. Nothing is now wanted for completing the solution of the problem, except the reducing of the expression (C.) to a form fit for numerical calculation.

Investigation of the integral Q_0.

We have

$$Q_0 = \int_0^m \frac{e\,d\,x\,c^{-x}}{\Delta} = \int_0^m \frac{e\,d\,x\,c^{-x}}{\sqrt{(1-e^2)^2 + 4\,e^2 \cdot \frac{x}{m}}}:$$

assume,

$$\Delta = \sqrt{(1-e^2)^2 + 4\,e^2 \cdot \frac{x}{m}} = 1 - e^2 + 2\,e^2\,z;$$

then,

$$\frac{e\,d\,x}{\Delta} = e \cdot m\,d\,z,$$

$$\frac{x}{m} = t = 1 - e^2\,(z - z^2).$$

By Lagrange's theorem,

$$\Psi = t - t^2,$$

$$z = t + e^2 \cdot \Psi + \frac{e^4}{1 \cdot 2} \cdot \frac{d \cdot \Psi^2}{d\,t} + \frac{e^6}{1 \cdot 2 \cdot 3} \cdot \frac{d\,d \cdot \Psi^3}{d\,t^2} + \&c.;$$

$$m\,d\,z = m\,d\,t \cdot \left\{1 + e^2 \cdot \frac{d \cdot \Psi}{d\,t} + \frac{e^4}{1 \cdot 2} \cdot \frac{d^2\,\Psi^2}{d\,t^2} + \frac{e^6}{1 \cdot 2 \cdot 3} \cdot \frac{d^3\,\Psi^3}{d\,t^3} + \&c.\right\};$$

consequently,

$$\int_0^m \frac{e\,d\,x\,c^{-x}}{\Delta} = \int_0^1 m\,d\,t\,c^{-mt}\left\{e + e^3\,\frac{d\Psi}{d\,t} + \frac{e^5}{1 \cdot 2} \cdot \frac{d^2\,\Psi^2}{d\,t^2} + \&c.\right\}.$$

Wherefore, if we assume

$$Q_0 = A_1\,e + A_3\,e^3 + A_5\,e^5 + \&c.,$$

* Précis Elem. de Physique, p. 229. tom. ii. edit. 2nd. Addit. à la Conn. des Temps, 1839, p. 36.

we shall have

$$A_{2n+1} = \frac{1}{1.2.3\ldots n} \int_0^1 m\,d\,t\,c^{-mt} \cdot \frac{d^n.\Psi^n}{d\,t^n}.$$

In the first place, it may be proper to show that all the coefficients in the series for Q^0 are positive. For this purpose integrate by parts, and the results will be

$$A_{2n+1} = \frac{m}{1.2.3\ldots n} \times \left\{ c^{-mt} \cdot \frac{d^{n-1}\Psi^n}{d\,t^{n-1}} + \int m\,d\,t\,c^{-mt} \cdot \frac{d^{n-1}\Psi^n}{d\,t^{n-1}} \right\}.$$

Now it is evident that

$$\frac{d^{n-1}\Psi^n}{d\,t^{n-1}}$$

is divisible both by t and $1 - t$: it is therefore zero at both the limits of the integral; so that we have simply

$$A_{2n+1} = \frac{m}{1.2.3\ldots n} \cdot \int_0^1 m\,d\,t\,c^{-mt} \cdot \frac{d^{n-1}\Psi^n}{d\,t^{n-1}}.$$

Continuing to integrate in like manner, we shall find after n successive operations,

$$A_{2n+1} = \frac{m^n}{1.2.3\ldots n} \int_0^1 m\,d\,t\,c^{-mt}\,\Psi^n,$$

which is obviously a positive quantity.

By expanding, we get

$$\Psi^n = t^n(1-t)^n = t^n - n.t^{n+1} + n \cdot \frac{n-}{2} \cdot t^{n+2} - \&c.:$$

and, by performing the differential operations,

$$\frac{1}{1.2.3\ldots n} \cdot \frac{d^n\Psi^n}{d\,t^n} = 1 - n.n+1 \cdot \frac{t}{1} + \left(n \cdot \frac{n-1}{2}\right) \cdot n+1.n+2 \cdot \frac{t^2}{1.2} - \&c.$$

Now, because $t = \frac{x}{m}$, if we put,

$$\Psi'(x) = \frac{1}{1.2.3\ldots n} \cdot \frac{d^n\Psi^n}{d\,t^n},$$

we shall have

$$\Psi'(x) = 1 - n \cdot \frac{n+1}{m} \cdot \frac{x}{1} + n \cdot \frac{n-1}{2} \cdot \frac{n+1.n+2}{m^2} \cdot \frac{x^2}{1.2} - \&c.$$

Another form may be given to this function; for, without any variation in quantity, t and $1 - t$ may be interchanged, not only in

$$\Psi^n = t^n(1-t)^n,$$

but in all its differentials, observing that the results equal in quantity will have opposite signs when the number of differentiations is odd, and the same sign when the number is even. Now if, instead of $t = \frac{x}{m}$, we substitute $1 - t = \frac{m-x}{m}$, we shall have

$$\Psi'(x) = \pm \left\{ 1 - n \cdot \frac{n+1}{m} \cdot \frac{m-x}{1} + n \cdot \frac{n-1}{2} \cdot \frac{n+1.n+2}{m^2} \cdot \frac{(m-x)^2}{1.2} - \&c. \right.$$

The coefficient A_{2n+1} is thus expressed in terms of x:

$$A_{2n+1} = \int_0^m dx\, c^{-x}\,\Psi'(x):$$

the indefinite integral is

$$-c^{-x}.\left\{\Psi'(x) + \frac{d.\Psi'(x)}{dx} + \frac{dd.\Psi'(x)}{dx^2} + \&c.\right\}.$$

This integral, taken between the limits $x = 0$ and $x = m$, is equal to A_{2n+1}: the first form of $\Psi'(x)$ will give the values of all the differentials at the limit $x = 0$; and the second form of the same function will give the like values at the other limit $x = m$: Thus we obtain,

$$A_{2n+1}\left\{1 - n\cdot\frac{n+1}{m} + n\cdot\frac{n-1}{2}\cdot\frac{n+1\,.\,n+2}{m^2} - \&c.\right\}$$
$$\mp c^{-m}.\left\{1 + n\cdot\frac{n+1}{m} + n\cdot\frac{n-1}{2}\cdot\frac{n+1\,.\,n+2}{m^2} + \&c.\right\},$$

the upper or lower sign taking place according as n is even or odd.

The numerical coefficients, computed by the formula, are as follows:

$$c^{-m} = c^{-10} = \cdot 0000454$$

$$A_1 = 1 - c^{-m} = 0\cdot 9999546$$

$$A_3 = \frac{4}{5} + \frac{6}{5}c^{-m} = 0\cdot 8000545$$

$$A_5 = \frac{13}{25} - \frac{43}{25}c^{-m} = 0\cdot 5199219$$

$$A_7 = \frac{7}{25} + \frac{73}{25}c^{-m} = 0\cdot 2801326$$

$$A_9 = \frac{16}{125} - \frac{726}{125}c^{-m} = 0\cdot 1277363$$

$$A_{11} = \frac{31}{625} + \frac{8359}{625}c^{-m} = 0\cdot 0502072$$

$$A_{13} = 0\cdot 0172805$$

$$A_{15} = 0\cdot 0052779$$

$$A_{17} = 0\cdot 0014467$$

$$A_{19} = 0\cdot 0003593$$

$$A_{21} = 0\cdot 0000815$$

$$A_{23} = 0\cdot 0000170$$

$$A_{25} = 0\cdot 0000036.$$

The horizontal refraction answers to $\cos\theta = 0$, $e = 1$; and the part of it depending on Q_0 is found by adding all the coefficients, viz.

$$\frac{\alpha(1+\alpha)}{\sqrt{5i}} \times 2\cdot 8024736 = 2036''\cdot 52.$$

If we take the integral between the limits $x = 0$, $x = \infty$, the result is not sensibly different, viz.

$$\alpha(1+\alpha)\int_0^{\infty} \frac{d\,x\,c^{-x}}{\sqrt{2\,i\,x}} = \frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2\,i}} = 2036''{\cdot}52.$$

Investigation of $\lambda \times Q_1$.

For this purpose we must find the value of

$$\int_0^m \frac{2\,d\,x\,c^{-2x}}{\Delta} = \int_0^m \frac{2\,d\,x\,c^{-2x}}{\sqrt{(1-e^2)^2 + 4\,e^2 \cdot \frac{x}{m}}} = \int_0^m \frac{2\,d\,x\,c^{-2x}}{\sqrt{(1-e^2)^2 + 4\,e^2 \cdot \frac{2\,x}{2\,m}}}:$$

this integral has therefore the same form as Q_0, the quantities $2\,x$ and $2\,m$ taking the place of x and m. Wherefore, if we assume

$$e \times \int \frac{2\,d\,x\,c^{-2x}}{\Delta} = a_1\,e + a_3\,e^3 + a_5\,e^5 + \&c.,$$

the value of a_{2n+1} will be found merely by writing $2\,m$ for m in the expression of A_{2n+1}; but as $c^{-2m} = c^{-20}$ is extremely minute, the part multiplied by it may be neglected. Thus,

$$a_{2n+1} = 1 - n \cdot \frac{n+1}{2\,m} + n \cdot \frac{n-1}{2} \cdot \frac{n+1\,.\,n+2}{(2\,m)^2} - \&c.$$

The numerical coefficients are as follows:

$$\begin{aligned}
a_1 &= 1\\
a_3 &= 0{\cdot}9\\
a_5 &= 0{\cdot}73\\
a_7 &= 0{\cdot}535\\
a_9 &= 0{\cdot}3555\\
a_{11} &= 0{\cdot}21505\\
a_{13} &= 0{\cdot}118945\\
a_{15} &= 0{\cdot}0604215\\
a_{17} &= 0{\cdot}0283127\\
a_{19} &= 0{\cdot}0122898\\
a_{21} &= 0{\cdot}0049621\\
a_{23} &= 0{\cdot}0018695\\
a_{25} &= 0{\cdot}0006623.
\end{aligned}$$

These values being found, if we assume

$$\lambda \times Q_1 = B_3\,e^3 + B_5\,e^5 + B_7\,e^7 + \&c.,$$

the term $\lambda \times c^{-m} \times e$, which is insensible, being omitted, we shall have

$$B_3 = \lambda\,(a_3 - A_3) = 0{\cdot}021866$$

$$\begin{aligned}
B_5 &= \lambda\ (a_5 - A_5) = 0{\cdot}045961\\
B_7 &= \lambda\ (a_7 - A_7) = 0{\cdot}055760\\
B_9 &= \lambda\ (a_9 - A_9) = 0{\cdot}049829\\
B_{11} &= \lambda\ (a_{11} - A_{11}) = 0{\cdot}036064\\
B_{13} &= \lambda\ (a_{13} - A_{13}) = 0{\cdot}022242\\
B_{15} &= \lambda\ (a_{15} - A_{15}) = 0{\cdot}012064\\
B_{17} &= \lambda\ (a_{17} - A_{17}) = 0{\cdot}005878\\
B_{19} &= \lambda\ (a_{19} - A_{19}) = 0{\cdot}002610\\
B_{21} &= \lambda\ (a_{21} - A_{21}) = 0{\cdot}001067\\
B_{23} &= \lambda\ (a_{23} - A_{23}) = 0{\cdot}000405\\
B_{25} &= \lambda\ (a_{25} - A_{25}) = 0{\cdot}000144.
\end{aligned}$$

By making $\cos\theta = 0$, $e = 1$, we shall have, for the approximate value of the part of the horizontal refraction depending on $\lambda\, Q_1$,

$$\frac{\alpha(1+\alpha)}{\sqrt{5i}} \times 0{\cdot}253891 = 184''{\cdot}50.$$

If the integrals be taken from $x = 0$ to $x = \infty$, the same quantity will be

$$\lambda \times \alpha(1+\alpha) \int \frac{dx\,(2c^{-2x} - c^{-x})}{\sqrt{2ix}} = \frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}} \cdot \lambda\,(\sqrt{2} - 1) = 184''{\cdot}56.$$

Between the two limits, the exact quantity obtained by integrating from $x = 0$ to $x = m = 10$, must lie; so that the error of the series is of no account.

It may be proper to make an observation here, which applies generally to the kind of integrals peculiar to this investigation. The first term of $\lambda\, Q_1$, viz. $B_1 e = \lambda\, c^{-m}\,.\,e$, which is rejected, varies with the height of the atmosphere. If a small number be taken for m, that is, in low atmospheres, the refractions will vary with the height, and will not agree with the observed quantities; if a considerable number be taken, as eight or ten, or any greater number, that is, if the atmosphere extend forty or fifty miles or more above the earth's surface, the refractions will not be sensibly different from what they would be in an atmosphere of unlimited height. The invariability of the refractions concurs with other phenomena to prove that the air reaches an elevation of fifty miles, more or less.

Investigation of $f \times Q_2$.

We have

$$\frac{Q_2}{e} = \int_0^m \frac{dx}{\Delta}\left(4c^{-2x} - 3c^{-x} + xc^{-x}\right).$$

Now the following formula is easily proved by differentiating,

$$\int \frac{dx}{\Delta} x c^{-x} = \frac{1}{2}\int \frac{dx\, c^{-x}}{\Delta} - \frac{m}{4} \cdot \frac{(1-e^2)^2}{e^2} \cdot \int \frac{dx\, c^{-x}}{\Delta} + \frac{m}{4} \cdot \frac{1 - e^2 - c^{-x}\Delta}{e^2},$$

all the integrals vanishing when $x = 0$. By extending the integrals to $x = m = 10$, in which case $\Delta = 1 + e^2$, the result will be

$$\int_0^m \frac{dx}{\Delta} x c^{-x} = \frac{1}{2} \int_0^m \frac{dx\, c^{-x}}{\Delta} - \frac{5}{2} \cdot \frac{(1-e^2)^2}{e^2} \cdot \int_0^m \frac{dx\, c^{-x}}{\Delta} + \frac{5}{2} \cdot \frac{1 - e^2 + c^{-m}(1+e^2)}{e^2}:$$

and, by substituting this value, we shall have

$$Q_2 = 2e \int_0^m \frac{2\,dx\,c^{-2x}}{\Delta} - \frac{5}{2} \int_0^m \frac{e\,dx\,c^{-x}}{\Delta} - \frac{5}{2} \cdot \frac{(1-e^2)^2}{e^2} \cdot \int_0^m \frac{e\,dx\,c^{-x}}{\Delta}$$
$$+ \frac{5}{2} \cdot \frac{1 - c^{-m}}{e} - \frac{5}{2}(1 + c^{-m})\, e.$$

The value of Q_2 will now be obtained in a series of the powers of e by putting for the integrals the equivalent series that have already been investigated. When this is done, the three first terms will be as follows:

$$\frac{5}{2}\left(1 - c^{-m} - A_1\right) \cdot \frac{1}{e}$$
$$+ \left(2\,a_1 - \frac{5}{2} A_1 + 5 A_1 - \frac{5}{2} A_3 - \frac{5}{2}(1 + c^{-m})\right) \cdot e$$
$$+ \left(2\,a_3 - \frac{5}{2} A_3 - \frac{5}{2} A_1 + 5 A_3 - \frac{5}{2} A_5\right) \cdot e^3.$$

Upon substituting the exact values of A_1, A_3, &c., the first of these terms is zero: the other two are as follows:

$$- 8\, c^{-m} \times e$$
$$+ \frac{49}{5} c^{-m} \times e^3;$$

the amount of which is very small even at the horizon; and, when multiplied by $f = \frac{2}{9}$, it becomes insensible. These terms being neglected, we may assume

$$Q_2 = C_5\, e^5 + C_7\, e^7 + C_9\, e^9 + \&c.;$$

and we shall find

$$C_5 = 2\,a_5 - \frac{5}{2} A_5 - \frac{5}{2} \Delta^2 A_3$$
$$C_7 = 2\,a_7 - \frac{5}{2} A_7 - \frac{5}{2} \Delta^2 A_5$$
$$\vdots$$
$$C_{2n+1} = 2\,a_{2n+1} - \frac{5}{2} A_{2n+1} - \frac{5}{2} \Delta^2 A_{2n-1}.$$

The numerical coefficients will now be obtained:

$\Delta^2 A_1 = -\cdot 0802325$

$\Delta^2 A_3 = +\cdot 0403433 \ldots\ldots C_5 = \cdot 059337$

$\Delta^2 A_5 =$	$\cdot 0873930$	$C_7 = \cdot 151186$
$\Delta^2 A_7 =$	$\cdot 0748672$	$C_9 = \cdot 204491$
$\Delta^2 A_9 =$	$\cdot 0446024$	$C_{11} = \cdot 193076$
$\Delta^2 A_{11} =$	$\cdot 0209241$	$C_{13} = \cdot 142381$
$\Delta^2 A_{13} =$	$\cdot 0081714$	$C_{15} = \cdot 087220$
$\Delta^2 A_{15} =$	$\cdot 0027438$	$C_{17} = \cdot 046149$
$\Delta^2 A_{17} =$	$\cdot 0008096$	$C_{19} = \cdot 021658$
$\Delta^2 A_{19} =$	$\cdot 0002133$	$C_{21} = \cdot 009187$
$\Delta^2 A_{21} =$	$\cdot 0000511$	$C_{23} = \cdot 003569$
$\Delta_2 A_{23} =$	$\cdot 0000105$	$C_{25} = \cdot 001290.$

As the value of f is not fixed with the same certainty as that of λ, the coefficients of Q_2 have not been multiplied by f: the intention of which is to make it more easy to determine a variation of the refraction, viz $\delta f \times Q_2$, answering to δf any variation of f that good observations may require.

The part of the horizontal refraction depending on Q_2 is

$$\frac{2}{9} \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times 0\cdot 919534 = 148''\cdot 51.$$

If we integrate the original expression of Q_2 from $x = 0$ to $x = \infty$, e being 1, we shall have

$$\frac{2}{9} \times \alpha(1+\alpha) \times \int \frac{dx\,(4c^{-2x} - 3c^{-x} + xc^{-x})}{\sqrt{2ix}} = \frac{2}{9} \cdot \frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}} \cdot \left(2\sqrt{2} - \frac{5}{2}\right) = 148''\cdot 63.$$

It thus appears that the error is less than $0''\cdot 12$; for the exact integral from $x = 0$ to $x = m = 10$, is less than the second number, and greater than the first on account of the terms of the series left out.

The next point that should engage attention is to find the value of $f' \times Q_3$. In the present state of our knowledge of the phenomena of the atmosphere, it seems impossible to determine f' by experiments. The probability is, that it is much less than f or $\frac{2}{9}$; and as the integral Q_3 is inconsiderable except within a degree or two above the horizon, and even at such low altitudes is not great; it follows that the part of the refraction depending on $f'\, Q_3$ will only be sensible, if at all, when a star is distant 88° or more from the zenith. At present the probability is, that there is no other way of ascertaining the value of f' but by good observed refractions at great distances from the zenith; which observations are neither numerous nor easily collected. From the uncertainty of the term $f' \times Q_3$, it cannot be estimated in constructing a table of mean refractions, which must therefore be deduced entirely from the other three terms, as in the paper of 1823. In this manner has the table in this paper been computed, by means of the formulas now to be explained. But the term $f'\, Q_3$ will after-

wards be discussed, and its value investigated, in order that it may be taken into account, if this should be found necessary, in the progressive improvement of the theory.

When the term $f' Q_3$ is left out, the expression of the refraction will be

$$\delta \theta = \sin \theta \times \frac{\alpha (1 + \alpha)}{\sqrt{5 i}} \cdot (Q_0 + \lambda Q_1 - f Q_2):$$

and if the equivalent series be substituted for the first two terms, and the series for Q_2 be multiplied by $f = \frac{2}{9}$, the result will be

$$\delta \theta = \sin \theta \times \frac{\alpha (1 + \alpha)}{\sqrt{5 i}} \times \left\{ \begin{array}{l} e \\ + 0{\cdot}821921 \,.\, e^3 \\ + 0{\cdot}552697 \,.\, e^5 \\ + 0{\cdot}302296 \,.\, e^7 \\ + 0{\cdot}132123 \,.\, e^9 \\ + 0{\cdot}043365 \cdot e^{11} \\ + 0{\cdot}007883 \,.\, e^{13} \\ - 0{\cdot}002040 \,.\, e^{15} \\ - 0{\cdot}002930 \,.\, e^{17} \\ - 0{\cdot}001842 \,.\, e^{19} \\ - 0{\cdot}000893 \,.\, e^{21} \\ - 0{\cdot}000371 \,.\, e^{23} \\ - 0{\cdot}000139 \,.\, e^{25} \end{array} \right.$$

To bring this formula to a form more convenient for calculation, all the coefficients must be reduced to seconds. The negative terms are all very small, never amounting to so much as $6''$, and of no account whatever, except the apparent altitude be equal to 2° or less; it will therefore be proper to separate these terms from the rest, representing their sum by the symbol V (θ). These things being attended to, we have, in the first place, this formula for computing e, viz.

$$\log . \tan \phi = \log . \sec . \theta + 19{\cdot}2067840 - 20: \; e = \tan \frac{\phi}{2}.$$

Next, reducing the arcs to seconds,

$$\frac{\alpha (1 + \alpha)}{\sqrt{5 i}} = 726''{\cdot}687:$$

$$\delta \theta = \sin \theta \times \left\{ \begin{array}{lll} e & \times 726''{\cdot}687, & \overset{\text{log.}}{2{\cdot}8613472} \\ + e^3 & \times 597{\cdot}280, & 2{\cdot}7761772 \\ + e^5 & \times 401{\cdot}638, & 2{\cdot}6038343 \end{array} \right.$$

$$
\begin{array}{lll}
+ e^{7} \times 219{\cdot}674, & 2{\cdot}3417796 \\
+ e^{9} \times 96{\cdot}012, & 1{\cdot}9823255 \\
+ e^{11} \times 31{\cdot}513, & 1{\cdot}4984866 \\
+ e^{13} \times 5{\cdot}728, & 0{\cdot}7580287 \\
- \mathrm{V}(\theta). &
\end{array}
$$

$$
\begin{array}{lll}
\mathrm{V}(\theta) = \sin\theta \times \Big\{ & e^{15} \times \overset{\prime\prime}{1}{\cdot}483, & \overset{\text{log.}}{0{\cdot}1710} \\
& + e^{17} \times 2{\cdot}129, & 0{\cdot}3282 \\
& + e^{19} \times 1{\cdot}337, & 0{\cdot}1266 \\
& + e^{21} \times 0{\cdot}649, & -1{\cdot}8122 \\
& + e^{23} \quad 0{\cdot}270, & -1{\cdot}4307 \\
& + e^{25} \quad 0{\cdot}102, & -1{\cdot}0072
\end{array}
$$

When $\theta = 87°$, V (θ) is zero; and if this function be computed for every succeeding half-degree, the quantity answering to any intermediate value of θ will be found by an easy interpolation. Such is the intention of the following Table; by the help of which any refraction from the zenith to the horizon may be computed by a series of the simplest form, and consisting of no more than seven terms.

θ.	V (δ θ.)
$87\frac{1}{2}$	$\overset{\prime\prime}{0}{\cdot}06$
88	0·14
$88\frac{1}{2}$	0·38
89	0·86
$89\frac{1}{2}$	2·30
90	5·97

If $e = 1$, the result will be the horizontal refraction, viz.

$$2078''{\cdot}53 - 5''{\cdot}97 = 2072''{\cdot}56,$$

which is almost exactly the same with $2072''{\cdot}46$, the quantity before computed in § 10 by a very different method.

12. We next proceed to inquire into the influence which the term multiplied by f', before omitted, may have on the refractions.

Investigation of the integral Q_3.

The expression of this integral is,

$$Q_3 = \int_0^m \frac{e\,dx}{\Delta}\left(8\,c^{-2x} - 8\,c^{-x} + 7\,x\,c^{-x} - 2\,x^2\,c^{-x} + \frac{x^3}{6}\,c^{-x}\right),$$

which is a negative quantity, as appears from the valuation of it in § 9: it will therefore contribute to distinctness if its sign be changed, in which case it will be thus

written,

$$Q_3 = \int_0^m \frac{e\,dx}{\Delta}\left(-8\,c^{-2x} + 8\,c^{-x} - 7\,x\,c^{-x} + 2\,x^2\,c^{-x} - \frac{x^3}{6}\,c^{-x}\right);$$

and the formula for the refractions will now be,

$$\delta\,\theta = \sin\theta \times \frac{\alpha\,(1+\alpha)}{\sqrt{5\,i}}\left(Q_0 + \lambda\,Q_1 - f\,Q_2 - f'\,Q_3\right).$$

Suppressing the tedious operations of reducing, we may put the integral Q_3, taken indefinitely, in the following form, which it is not difficult to verify by differentiating:

$$\varepsilon = \frac{1-e^2}{e},$$

$$\begin{aligned} Q_3 = &- 4\int\frac{e\,.\,2\,d\,x\,c^{-2x}}{\Delta} + \frac{91}{16}\int\frac{e\,d\,x\,c^{-x}}{\Delta} \\ &+ \left(\frac{215}{16}\,\varepsilon^2 + \frac{175}{16}\,\varepsilon^4 + \frac{125}{48}\,\varepsilon^6\right).\int\frac{e\,d\,x\,c^{-x}}{\Delta} \\ &+ \frac{c^{-x}\Delta}{e}\left(\frac{185}{16} + \frac{125}{12}\,\varepsilon^2 + \frac{125}{48}\,\varepsilon^4\right) \\ &- \varepsilon\left(\frac{185}{16} + \frac{125}{12}\,\varepsilon^2 + \frac{125}{48}\,\varepsilon^4\right) \\ &- \frac{c^{-x}\Delta}{e}\left(\frac{95}{24}\,x - \frac{5}{12}\,x^2 + \frac{25}{24}\,x\,.\,\varepsilon^2\right). \end{aligned}$$

This being the indefinite integral, the value of Q_3 in the formula for the refractions will be obtained by putting $x = m = 10$; which gives

$$\frac{c^{-x}\Delta}{e} = \frac{1+e^2}{e}\cdot c^{-m}:$$

and this value, as well as that of ε, being substituted, the quantity sought will be expressed as follows:

$$\begin{aligned} Q_3 = &- 4\int\frac{e\,.\,2\,d\,x\,c^{-2x}}{\Delta} + \frac{91}{16}\int\frac{e\,d\,x\,c^{-x}}{\Delta} \\ &+ \left\{\frac{215}{16}\left(\frac{1-e^2}{e}\right)^2 + \frac{175}{16}\left(\frac{1-e^2}{e}\right)^4 + \frac{125}{48}\left(\frac{1-e^2}{e}\right)^6\right\}.\int\frac{e\,d\,x\,c^{-x}}{\Delta} \\ &+ c^{-m}\left(\frac{125}{48}\,\frac{1}{e^5} - \frac{125}{16}\,\frac{1}{e^3} + \frac{905}{48}\cdot\frac{1}{e} + \frac{905}{48}\,e - \frac{125}{16}\,e^3 + \frac{127}{48}\,e^5.\right) \\ &- \frac{125}{48}\cdot\frac{1}{e^5} + \frac{125}{48}\,\frac{1}{e^3} - \frac{305}{48}\,\frac{1}{e} + \frac{305}{48}\,e - \frac{125}{48}\,e^3 + \frac{125}{48}\,e^5. \end{aligned}$$

The series equivalent to the integrals must now be substituted, in order to express the quantity sought in terms containing the powers of e.

In the first place we have these three terms, each of which is zero when the exact values of A_1, A_3, &c. are substituted, viz.

$$\frac{125}{48}\,(A_1 - 1 + c^{-m})\cdot\frac{1}{e}$$

$$+\left\{\frac{175}{16}A_1+\frac{125}{48}(A_3-6A_1)+\frac{125}{48}-\frac{125}{16}c^{-m}\right\}.\frac{1}{e^3}$$

$$+\left\{\frac{215}{16}A_1+\frac{175}{16}(A_3-4A_1)+\frac{125}{48}(A_5-6A_3+15A_1)-\frac{305}{48}+\frac{905}{48}c^{-m}\right\}\times\frac{1}{e^5}$$

The next three terms are as follows:

$$\left\{-4a_1+\frac{91}{16}A_1+\frac{215}{16}(A_3-2A_1)+\frac{175}{16}(A_5-4A_3+6A_1)\right.$$
$$\left.+\frac{125}{48}(A_7-6A_5+15A_3-20A_1)+\frac{305}{48}+\frac{905}{48}c^{-m}\right\}.e$$

$$+\left\{-4a_3+\frac{91}{16}A_3+\frac{215}{16}(A_5-2A_3+A_1)\right.$$
$$+\frac{175}{16}(A_7-4A_5+6A_3-4A_1)$$
$$+\frac{125}{48}(A_9-6A_7+15A_5-20A_3+15A_1)$$
$$\left.-\frac{125}{48}-\frac{125}{16}c^{-m}\right\}.e^3$$

$$+\left\{-4a_5+\frac{91}{16}A_5+\frac{215}{16}(A_7-2A_5+A_3)\right.$$
$$+\frac{175}{16}(A_9-4A_7+6A_5-4A_3+A_1)$$
$$+\frac{125}{48}(A_{11}-6A_9+15A_7-20A_5+15A_3-6A_1)$$
$$\left.+\frac{125}{48}+\frac{125}{48}c^{-m}\right\}.e^5.$$

On substituting the exact values of A_1, A_3, &c., these three terms will come out as follows:

$$+\frac{158}{3}c^{-m}.e,\qquad \text{or } +\cdot 00239\,.\,e$$

$$-\frac{348}{5}c^{-m}.e^3,\qquad \text{or } -\cdot 00316\,.\,e^3$$

$$+\frac{8891}{75}c^{-m}.e^5,\qquad \text{or } +\cdot 00538\,.\,e^5.$$

These three terms are the part of the refraction that depends on the height of the atmosphere: at the horizon, or when $e=1$, their amount is greatest and equal to

$$f'\times\frac{\alpha(1+\alpha)}{\sqrt{5i}}\times\cdot 00461=f'\times 726''\cdot 7\times\cdot 00461=f'\times 3''\cdot 3,$$

which, on account of the smallness of f', will be a minute fraction of a second.

Rejecting the six foregoing terms, we may assume

$$Q_3=H_7e^7+H_9e^9+H_{11}e^{11}+\&c.:$$

and, having computed the differences in the following table,

	Δ^2	Δ^4	Δ^6
A_1	....	....	+·0278859
A_3	....	—·0595755	—·0175110
A_5	+·0873930	—·0177390	—·0199864
A_7	+·0748672	+·0065865	—·0079396
A_9	+·0446024	+·0109256	—·0002312
A_{11}	+·0209241	+·0073251	+·0016762
A_{13}	+·0081714	+·0034934	+·0012515
A_{15}	+·0027438	+·0013379	+·0005925
A_{17}	+·0008096	+·0004339	+·0001891
A_{19}	+·0002133	+·0001224	....
A_{21}	+·0000509	....	....

we shall have

$$H_7 = -4\,a_7 + \frac{91}{16}A_7 + \frac{215}{16}\Delta^2 A_5 + \frac{175}{16}\Delta^4 A_3 + \frac{125}{48}\Delta^6 A_1 = \cdot 04861$$

$$H_9 = -4\,a_9 + \frac{91}{16}A_9 + \frac{215}{16}\Delta^2 A_7 + \frac{175}{16}\Delta^4 A_5 + \frac{125}{48}\Delta^6 A_3 = \cdot 07091$$

$$H_{11} = -4\,a_{11} + \frac{91}{16}A_{11} + \frac{215}{16}\Delta^2 A_9 + \frac{175}{16}\Delta^4 A_7 + \frac{125}{48}\Delta^6 A_5 = \cdot 04469$$

$$H_{13} = -4\,a_{13} + \frac{91}{16}A_{13} + \frac{215}{16}\Delta^2 A_{11} + \frac{175}{16}\Delta^4 A_9 + \frac{125}{48}\Delta^6 A_7 = \cdot 00249$$

$$H_{15} = -4\,a_{15} + \frac{91}{16}A_{15} + \frac{215}{16}\Delta^2 A_{13} + \frac{175}{16}\Delta^4 A_{11} + \frac{125}{48}\Delta^6 A_9 = -\cdot 02230$$

$$H_{17} = -4\,a_{17} + \frac{91}{16}A_{17} + \frac{215}{16}\Delta^2 A_{15} + \frac{175}{16}\Delta^4 A_{13} + \frac{125}{48}\Delta^6 A_{11} = -\cdot 02558$$

$$H_{19} = -4\,a_{19} + \frac{91}{16}A_{19} + \frac{215}{16}\Delta^2 A_{17} + \frac{175}{16}\Delta^4 A_{15} + \frac{125}{48}\Delta^6 A_{13} = -\cdot 01835$$

$$H_{21} = -4\,a_{21} + \frac{91}{16}A_{21} + \frac{215}{16}\Delta^2 A_{19} + \frac{175}{16}\Delta^4 A_{17} + \frac{125}{48}\Delta^6 A_{15} = -\cdot 01023$$

$$H_{23} = -4\,a_{23} + \frac{91}{16}A_{23} + \frac{215}{16}\Delta^2 A_{21} + \frac{175}{16}\Delta^4 A_{19} + \frac{125}{48}\Delta^6 A_{17} = -\cdot 00487.$$

The coefficients of the assumed series being found, and being expressed in seconds of a degree, the part of the refractions depending on Q_3 will be as follows:

$$f' \times \sin\theta \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times Q_3 = f' \times \sin\theta \times \left\{ \begin{array}{ll} & \text{log.} \\ e^7 \times 35''\!\cdot\!324, & 1\cdot 54807 \\ + e^9 \times 51\cdot 529, & 1.71205 \\ + e^{11} \times 32\cdot 476, & 1\cdot 51156 \\ + e^{13} \times 1\cdot 809, & 0\cdot 25755 \\ - e^{15} \times 16\cdot 205, & 1\cdot 20965 \\ - e^{17} \times 18\cdot 588, & 1\cdot 26925 \end{array} \right.$$

$$\left.\begin{array}{lll} & '' & \text{log.} \\ - e^{19} \times & 13{\cdot}334, & 1{\cdot}12498 \\ - e^{21} \times & 7{\cdot}427, & 0{\cdot}87080 \\ - e^{23} \times & 3{\cdot}480, & 0{\cdot}54158 \end{array}\right\}$$

The amount of this expression at the horizon, or when $e = 1$, is $f' \times 62''{\cdot}1$, almost the same with $f' \times 62''{\cdot}4$, which, as is shown in § 9, is the limit of the integral when it is extended from $x = 0$ to $x = \infty$. It is thus proved that the error of the series is of no account. This part of the refraction cannot be computed because f' is unknown. But although the precise value of f' is uncertain, it is probably very considerably less than f, or $\frac{2}{9}$; so that the effect on the refraction cannot exceed a few seconds even at the horizon. We shall be better able to form a just notion with respect to this point, when the Theoretical Table in this paper is compared with observations.

13. It remains to investigate the corrections that must be made in the practical application for the deviations indicated by the meteorological instruments from the mean constants used in constructing the table.

For this purpose we have

$$\delta\theta = \sin\theta \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times \text{S},$$

$$\text{S} = \text{Q}_0 + \lambda\,\text{Q}_1 - f\,\text{Q}_2,$$

$$\frac{\sqrt{5i}}{\cos\theta} = \frac{e}{1-e^2},$$

$$\lambda = \frac{\alpha}{i}.$$

The quantities e and λ depend only upon α and i: α varies both with the barometer and thermometer, and i, with the thermometer only: the quantity f does not seem liable to change in our climate. Admitting that the prefix d refers only to variations of the barometer and thermometer, we shall have

$$\delta\theta + d\,.\,\delta\theta = \sin\theta \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times \left\{ \left(1 + \frac{d\alpha}{\alpha} - \frac{1}{2}\cdot\frac{di}{i}\right).\,\text{S} + \frac{de}{e}\cdot\frac{d\text{S}}{de}\,e + \frac{d\lambda}{\lambda}\cdot\lambda\,\text{Q}_1 \right\}.$$

Now

$$\frac{de}{e} = \frac{1}{2}\cdot\frac{di}{i}\cdot\frac{1-e^2}{1+e^2},$$

$$\frac{d\lambda}{\lambda} = \frac{d\alpha}{\alpha} - \frac{di}{i}:$$

wherefore

$$\delta\theta + d.\delta\theta = \delta\theta\left(1 + \frac{d\alpha}{\alpha}\right)$$

$$- \sin\theta \cdot \frac{\alpha(1+\alpha)}{\sqrt{5i}} \cdot \frac{di}{i}\left(\frac{S}{2} - \frac{1}{2} \cdot \frac{1-e^2}{1+e^2} \cdot \frac{dS}{de}\, e\right)$$

$$+ \sin\theta \cdot \frac{\alpha(1+\alpha)}{\sqrt{5i}} \cdot \left(\frac{d\alpha}{\alpha} - \frac{di}{i}\right) . \lambda\, Q_1.$$

If p denote the observed height of the barometer, reduced to the fixed temperature of 50° of FAHR.; and τ the temperature of the air on the same scale; then, $\beta = \frac{1}{480}$,

$$1 + \frac{d\alpha}{\alpha} = \frac{1}{1 + \beta(\tau - 50)} \cdot \frac{p}{30},$$

$$\frac{d\alpha}{\alpha} = -\frac{\tau - 50}{480} - \frac{30 - p}{30},$$

$$\frac{di}{i} = +\frac{\tau - 50}{480},$$

$$\frac{d\alpha}{\alpha} - \frac{di}{i} = -2 \times \frac{\tau - 50}{480} - \frac{30 - p}{30}.$$

These values being found, if we put

$$T = \sin\theta \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times \frac{1}{480} \times \left(\frac{(1+e^2)S - (1-e^2)\frac{dS}{de}\, e}{2(1+e^2)} + 2\lambda\, Q_1\right),$$

$$b = \sin\theta \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times \frac{2\lambda\, Q_1}{30};$$

the expression of the mean refraction with its correction will be as follows,

$$\delta\theta + d.\delta\theta = \frac{\delta\theta}{1 - \beta(\tau - 50)} \cdot \frac{p}{30} - T.(\tau - 50) - b(30 - p).$$

The first term of this expression is the mean refraction corrected in the manner usually practised by Astronomers. If we assume that the temperature of the mercury in the barometer is the same with that of the air, this term will be equal to

$$\frac{1}{1 + \beta(\tau - 50)} \cdot \frac{1}{1 + \frac{\tau - 50}{10000}} \cdot \frac{p}{30} = \frac{1}{1 + c(\tau - 50)} \cdot \frac{p}{30},$$

$$c = \cdot 002183,$$

the new factor being added to compensate the expansion of the mercury. Two subsidiary tables are given for computing this part: Table II. contains the logarithms of $\frac{1}{1 + c(\tau - 50)}$ for 30° on either side of the mean temperature 50°, negative indices being avoided by substituting the arithmetical complements; and Table III. contains the logarithms, or the arithmetical complements, for all values of p from 31 to 28.

The coefficients, T and b, of the other two terms vary with the distance from the

zenith; and they can be computed in no other way than by reducing them to series of the powers of e. By substituting for $\lambda\, Q_1$, the equivalent series already known, we immediately obtain

$$b = \sin\theta \cdot \frac{\alpha(1+\alpha)}{\sqrt{5i}} \cdot \frac{1}{30} \cdot \left\{ B_3\, e_3 + B_5\, e^5 + B_7\, e^7 + \&c. \right\}.$$

Further, by expanding S and its differential, the expression of T will take this form,

$$T = \sin\theta \cdot \frac{\alpha(1+\alpha)}{\sqrt{5i}} \cdot \frac{1}{480} \cdot \left\{ G_3\, e^3 + G_7\, e^7 + G_9\, e^9 + \&c. \right\};$$

and we shall have

$$G_3 \;= A_1 - A_3 + 2\,B_3 = 0{\cdot}2436$$
$$G_5 \;= -A_1 + 3\,A_3 - 2\,A_5 + 2\,B_5 = 0{\cdot}4523$$
$$G_7 \;= A_1 - 3\,A_3 + 5\,A_5 - 3\,A_7 + 2\,B_7 = 0{\cdot}4705$$
$$G_9 \;= -A_1 + 3\,A_3 - 5\,A_5 + 7\,A_7 - 4\,A_9 + 2\,B_9 = 0{\cdot}3502$$
$$G_{11} = A_1 - 3\,A_3 + 5\,A_5 - 7\,A_7 + 9\,A_9 - 5\,A_{11} + 2\,B_{11} = 0{\cdot}2092$$
$$G_{13} = -A_1 + 3\,A_3 - 5\,A_5 + 7\,A_7 - 9\,A_9 + 11\,A_{11} - 6\,A_{13} + 2\,B_{13} = 0{\cdot}1050.$$

The series for T and b being now known, the coefficients of the terms must next be expressed in seconds of a degree; which being done, the following final results will be obtained.

		log.			log.
$T = \sin\theta \times \{ e^3$	$.\,0''{\cdot}369,$	$-\ 1{\cdot}5668,$	$b = \sin\theta \times \{ e^3$	$.\,0''{\cdot}530,$	$-\ 1{\cdot}7240$
$+\, e^5$	$.\,0{\cdot}685,$	$-\ 1{\cdot}8356$	$+\, e^5$	$.\,1{\cdot}113,$	$0{\cdot}0465$
$+\, e^7$	$.\,0{\cdot}712,$	$-\ 1{\cdot}8526$	$+\, e^7$	$.\,1{\cdot}350,$	$0{\cdot}1306$
$+\, e^9$	$.\,0{\cdot}530,$	$-\ 1{\cdot}7263$	$+\, e^9$	$.\,1{\cdot}207,$	$0{\cdot}0817$
$+\, e^{11}$	$.\,0{\cdot}317,$	$-\ 1{\cdot}5006$	$+\, e^{11}$	$.\,0{\cdot}873,$	$1{\cdot}9412$
$+\, e^{13}$	$.\,0{\cdot}159,$	$-\ 1{\cdot}2013$	$+\, e^{13}$	$.\,0{\cdot}539,$	$-\ 1{\cdot}7313$

The values of T and b are added in separate columns of the annexed table for altitudes less than 10°: for greater altitudes they are omitted as of no account. The application for finding the corrected refraction from the formula

$$\delta\theta + d.\delta\theta = \frac{\delta\theta}{1 + c(\tau - 50)} \cdot \frac{p}{30} - T(\tau - 50) - b(30 - p),$$

will best be explained by the examples afterwards given.

14. The Theoretical Table of refractions which has been computed by the foregoing formulas, and which is deduced solely from the phenomena of the atmosphere without arbitrary assumptions, is next to be compared with the tables most esteemed by astronomers. Two tables more eminently deserve this character; namely, BESSEL's table with its supplement in the *Tabulæ Regiomontanæ*, which may be considered as the result of observations, and as being nearly exact to 88° or 88°½ from the zenith; and

the table published annually in the *Connaissance des Temps*. As all the tables are supposed to contain the same series of refractions, the numbers corresponding to the same altitude should have constantly the same proportion: so that taking the number α which answers to the zenith-distance θ in BESSEL's table, the logarithm of the refraction at the same zenith distance in the New Table should be equal to

$$\log \alpha + \log \tan \theta + \cdot 00507,$$

the number ·00507 being the difference of the logarithms of the refractions at the altitude of 45° in the two tables: but, in the supplemental table, which contains the logarithms of the refractions, it is sufficient to add ·00507 to obtain the logarithms in the New Table. With regard to the refractions in the *Conn. des Temps*, it is more convenient to use the Table in the *Tables Astronomiques* published by the French Board of Longitude: for the logarithms in this table with the addition of ·0011, should agree respectively with the logarithms of the New Table. According to these directions the following comparative view has been drawn up.

Zenith dist.	Refractions.		
	New Table.	Tab. Reg.	Conn. des Temps.
10°	10″·30	10″·30	10″·30
20	21·26	21·26	21·26
30	33·72	33·72	33·72
40	48·99	48·99	48·99
45	58·36	58·36	58·36
50	69·52	69·52	69·52
55	83·25	83·24	83·25
60	100·85	100·85	100·86
65	124·65	124·62	124·65
70	159·16	159·11	159·22
75	214·70	214·58	214·83
80	320·19	319·88	320·63
81	353·79	353·38	354·33
82	394·68	394·20	395·37
83	445·42	444·86	445·87
84	509·86	509·23	511·22
85	593·96	593·38	595·80
85½	646·21	647·10	648·34
86	707·43	707·15	712·07
86½	779·92	777·36	783·07
87	866·76	864·59	870·37
87½	971·93	972·21	975·89
88	1101·35	1101·40	1105·1
88½	1262·6	1265·5	1265·0
89	1466·8	1481·8	1464·9
89½	1729·5	1764·9	1716·4

From this view it appears that the three Tables agree within less than 1″ as far as 80° from the zenith; the New Table is in accordance with BESSEL's, with slight discrepancies, to 88° or 88½° from the zenith; from 80° to 88° of zenith distance the numbers in the French Table exceed those in BESSEL's, the excess being 2″ at 84°, and 4″ at 88°. But when the distance from the zenith is greater than 80°, the accuracy of

the French Table is questionable, both on account of the hypothetical law of the densities, and because the quantity assumed for the horizontal refraction is uncertain.

A few examples are subjoined, as well for explaining the use of the New Table as for affording some indications of its accuracy at low altitudes. The two first instances are taken from the *Tables Astronomiques*, and are likewise published yearly in the *Conn. des Temps*.

EXAMPLE 1.

$\theta = 86^\circ\, 14'\, 42''$

Therm. $8^\circ{\cdot}75$ cent. $= 47^\circ{\cdot}75$ F.

Barom. $0^{m}{\cdot}741 = 29{\cdot}17$ in.

	86° 10′	2·86345
	4 42″	664
		2·87009
	Therm.	·00214
	Barom.	9·98781
Log $\delta\theta$		2·86004
$\delta\theta$		724·5
$-\cdot25 \times -2\frac{1}{4}$		$+\cdot5$
$-\cdot4 \times \cdot8$		$-\cdot3$
Corrected refraction		12′ 4″·7
Observed refraction		12 4 ·2

EXAMPLE 2.

$\theta = 86^\circ\, 15'\, 20''$

Therm. $8\frac{1}{2}^\circ$ cent. $= 46^\circ{\cdot}9$ F.

Barom. $0^{m}{\cdot}766 = 30{\cdot}16$ in.

	86° 10′	2·86345
	5 20″	753
		2·87098
	Therm.	·00276
	Barom.	·00232
Log $\delta\theta$		2·87606
$\delta\theta$		751·7
$-\cdot25 \times -3\cdot1$		$+\cdot8$
$-\cdot4 \times -\cdot16$		$+\cdot6$
Corrected refraction.		12′ 33″·1
Observed refraction		12 32 ·5

EXAMPLE 3.

Mean of 42 sub-polar observations of α Lyræ by Dr. BRINKLEY.

Irish Transactions, 1815.

$\theta = 87^\circ\, 42'\, 10''$

Therm. 35°

Barom. 29·5

	87° 40′	3·00522
	2 10″	392
		3·00914
	Therm.	·01444
	Barom.	9·99270
Log $\delta\theta$		3·01628
$\delta\theta$		1038″·2
$-\cdot6 \times -15$		$+9\cdot0$
$-1\cdot13 \times \frac{1}{2}$		$-0\cdot6$
Corrected refraction.		17′ 26″·6
Observed refraction.		17 26 ·5

EXAMPLE 4.

Mean of 10 observations of Capella, from a memoir of M. PLANA.

Acad. de Turin, tom. 32.

$\theta = 88^\circ\, 24'\, 9''{\cdot}7$

Therm. $47^\circ{\cdot}75$

Barom. 29·75

	88° 20′	3·08087
	4 9″·4	847
		3·08934
	Therm.	·00214
	Barom.	9·99607
Log $\delta\theta$		3·08755
$\delta\theta$		1223″·3
$-\cdot95 \times -2\frac{1}{4}$		$+2\cdot1$
$-1\cdot6 \times \cdot27$		$-0\cdot4$
Corrected refraction.		20′ 25″
Observed refraction.		20 24 ·3

We may now inquire how far the refractions are likely to be affected by the term which it was found necessary to leave out, because the present state of our know-

ledge of the phenomena of the atmosphere made it impossible to determine the coefficient f' by which it is multiplied. For this purpose the term alluded to, viz.

$$\sin\theta \times f' \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times Q_3,$$

which may be shortly denoted by $f' \times \chi(\theta)$, has been computed by means of the equivalent series, for every half degree between 85° and 88°, the results being as follows:

θ	$f' \times \chi(\theta)$
85°	$f' \times$ 1″·5
$85\frac{1}{2}$	$f' \times$ 2·0
86	$f' \times$ 3·3
$86\frac{1}{2}$	$f' \times$ 4·9
87	$f' \times$ 7·4
$87\frac{1}{2}$	$f' \times$ 11·2
88	$f' \times$ 17·0

From this view it appears that f', although considerably less than f or $\frac{2}{9}$ may still have some influence on the refractions at very low altitudes. The mean refraction in BESSEL's Table, and in the New Table, can hardly be supposed to differ 2″ from the true quantity, which would limit f' to be less than $\frac{1}{10}$. It is a matter of some importance to obtain a near value of f': and it is probable that this can be accomplished in no other way but by searching out such values of f and f' as will best represent many good observed refractions at altitudes less than 5°. If such values were found, our knowledge of the decrease of heat in ascending in the atmosphere would be improved, and the measurement of heights by the barometer would be made more perfect.

April 25*th*, 1838.

Table I.

Mean Refractions for the Temperature 50° Fahrenheit, and the barometric Pressure 30 inches.

Zenith dist.	δθ.	Log δθ.	Diff.	T.	C.	Zenith dist.	δθ.	Log δθ.	Diff.	T.	C.
1°	0′ 1″·02	0·0085	3012			53°	1′ 17″·38	1·88863	1577		
2	2·04	0·3097	1763			54	20·24	1·90440	1596		
3	3·06	0·4860	1252			55	23·25	1·92036	1617		
4	4·08	0·6112	974			56	26·41	1·93653	1638		
5	5·11	0·7086	796			57	29·73	1·95291	1664		
6	6·14	0·7882	675			58	33·23	1·96955	1691		
7	7·17	0·8557	587			59	36·93	1·98646	1722		
8	8·21	0·9144	519			60	40·85	2·00368	1756		
9	9·25	0·9663	466			61	45·01	2·02124	1794		
10	10·30	1·0129	424			62	49·44	2·03918	1836		
11	11·35	1·0553	388			63	54·17	2·05754	1881		
12	12·42	1·0941	359			64	59·23	2·07635	1932		
13	13·49	1·1300	334			65	2 4·65	2·09567	1988		
14	14·57	1·1634	313			66	10·48	2·11555	2048		
15	15·65	1·1947	294			67	16·78	2·13603	2116		
16	16·75	1·2241	278			68	23·61	2·15719	2191		
17	17·86	1·2519	265			69	31·04	2·17910	2275		
18	18·98	1·2784	252			70 00	39·16	2·20186	388		
19	20·11	1·3036	241			10	40·59	2·20573	390		
20	21·26	1·3277	230			20	42·04	2·20963	393		
21	22·42	1·3507	222			30	43·52	2·21356	396		
22	23·60	1·3729	215			40	45·02	2·21752	398		
23	24·80	1·3944	207			50	46·53	2·22150	402		
24	26·01	1·4151	201			71 00	48·08	2·22552	404		
25	27·24	1·4352	195			10	49·65	2·22956	407		
26	28·49	1·4547	189			20	51·25	2·23363	410		
27	29·75	1·4736	185			30	52·87	2·23773	413		
28	31·05	1·4921	181			40	54·53	2·24186	417		
29	32·38	1·5102	177			50	56·21	2·24603	419		
30	33·72	1·5279	173			72 00	57·92	2·25022	423		
31	35·09	1·5452	170			10	59·66	2·25445	425		
32	36·49	1·5622	168			20	3 1·43	2·25870	429		
33	37·93	1·5790	164			30	3·23	2·26299	433		
34	39·39	1·5954	162			40	5·06	2·26732	436		
35	40·89	1·6115	160			50	6·93	2·27168	440		
36	42·42	1·6276	159			73 00	8·83	2·27608	443		
37	44·00	1·6435	156			10	10·77	2·28051	447		
38	45·61	1·6591	155			20	12·74	2·28498	450		
39	47·27	1·6746	155			30	14·75	2·28948	454		
40	48·99	1·6901	154			40	16·80	2·29402	458		
41	50·75	1·7055	152			50	18·88	2·29860	462		
42	52·57	1·7207	151			74 00	21·01	2·30322	467		
43	54·43	1·7358	152			10	23·18	2·30789	470		
44	56·35	1·7510	151			20	25·39	2·31259	475		
45	58·36	1·76611	1512			30	27·66	2·31734	479		
46	1 0·43	1·78123	1514			40	29·95	2·32213	483		
47	2·57	1·79637	1518			50	32·30	2·32696	488		
48	4·80	1·81155	1523			75 00	34·70	2·33184	493		
49	7·11	1·82678	1530			10	37·16	2·33677	497		
50	9·52	1·84208	1539			20	39·65	2·34174	502		
51	12·02	1·85747	1551			30	42·21	2·34676	507		
52	14·64	1·87298	1565			40	44·82	2·35183	512		
53	17·38	1·88863				50	47·48	2·35695			

2 G 2

TABLE I. (Continued.)

Zenith dist.	δθ.	Log. δθ.	Diff.	T.	C.
75° 50	3′ 47″·48	2·35695	517		
76 00	50·21	2·36212	523		
10	53·00	2·36735	528		
20	55·85	2·37263	533		
30	58·76	2·37796	538		
40	4 1·74	2·38334	545		
50	4·79	2·38879	551		
77 00	7·91	2·39430	557		
10	11·11	2·39987	563		
20	14·39	2·40550	569		
30	17·74	2·41119	576		
40	21·19	2·41695	583		
50	24·72	2·42278	589		
78 00	28·33	2·42867	596		
10	32·04	2·43463	603		
20	35·84	2·44066	611		
30	39·75	2·44677	618		
40	43·76	2·44295	626		
50	47·88	2·45921	635		
79 00	52·12	2·46556	642		
10	56·47	2·47198	650		
20	5 0·94	2·47848	659		
30	5·54	2·48507	669		
40	10·28	2·49176	677		
50	15·16	2·49853	688	·03	·04
80 00	20·19	2·50541	696		
10	25·36	2·51237	707		
20	30·70	2·51944	716		
30	36·20	2·52660	727		
40	41·88	2·53387	738		
50	47·74	2·54125	749	·04	·05
81 00	53·79	2·54874	761		
10	6 0·04	2·55635	772		
20	6·50	2·56407	785		
30	13·18	2·57192	797		
40	20·09	2·57989	811		
50	27·26	2·58800	824	·05	·08
82 00	34·68	2·59624	838		
10	42·37	2·60462	851		
20	50·33	2·61313	866		
30	58·59	2·62179	883		
40	7 7·19	2·63062	899		
50	16·13	2·63961	916	·07	·11
83 00	25·42	2·64877			

Zenith dist.	δθ.	Log δθ.	Diff.	T.	C.
83° 00	7′ 25″·42	2·64877	833		
10	35·09	2·65810	948		
20	45·14	2·66758	970		
30	55·64	2·67728	985		
40	8 6·55	2·68713	1005		
50	17·95	2·69718	1028		
84 00	29·86	2·70746	1047	·10	·16
10	42·31	2·71793	1069		
20	55·33	2·72862	1092		
30	9 8·96	2·73954	1116		
40	23·25	2·75070	1140		
50	38·23	2·76210	1166		
85 00	53·96	2·77376	1194	·15	·24
10	10 10·52	2·78570	1219		
20	27·90	2·79789	1248		
30	46·21	2·81037	1281		
40	11 5·55	2·82318	1312		
50	25·90	2·83626	1342		
86 00	47·43	2·84968	1377	·24	·39
10	12 10·21	2·86345	1412		
20	34·34	2·87757	1448		
30	59·92	2·89205	1488	·31	·51
40	13 27·11	2·90693	1627		
50	55·99	2·92220	1570		
87 00	14 26·76	2·93790	1612	·39	·67
10	59·54	2·95402	1658	·43	·75
20	15 34·55	2·97060	1704	·47	·83
30	16 11·93	2·98764	1758	·52	·91
40	52·10	3·00522	1808	·58	1·01
50	17 35·12	3·02330	1862	·63	1·13
88 00	18 21·35	3·04192	1918	·69	1·24
10	19 11·07	3·06110	1977	·78	1·41
20	20 4·68	3·08087	2040	·87	1·58
30	21 2·60	3·10127	2102	·96	1·75
40	22 5·22	3·12229	2169	1·07	2·00
50	23 13·11	3·14398	2239	1·19	2·24
89 00	24 26·8	3·16637	2316	1·32	2·48
10	25 46·8	3·18943	2388	1·52	2·91
20	27 14·2	3·21331	2461	1·72	3·34
30	28 49·5	3·23792	2529	1·92	3·77
40	30 33·2	3·26321	2573	2·20	4·34
50	32 15·1	3·28894		2·48	5·00
90 00	34 32				

Table II.

Thermometer.					
	Log.	Diff.		Log.	Diff.
50°	0·00000		50°	0·00000	
49	0·00094		51	9·99906	
48	0·00190		52	9·99811	
47	0·00285		53	9·99717	
46	0·00380		54	9·99623	
45	0·00476	96	55	9·99529	
44	0·00572		56	9·99434	94
43	0·00668		57	9·99341	
42	0·00764		58	9·99248	
41	0·00861		59	9·99154	
40	0·00957		60	9·99061	
39	0·01053		61	9·98969	
38	0·01151	98	62	9·98875	
37	0·01248		63	9·98783	92
36	0·01346		64	9·98690	
35	0·01444		65	9·98598	
34	0·01541		66	9·98506	
33	0·01640		67	9·98414	
32	0·01738		68	9·98323	
31	0·01837		69	9·98231	
30	0·01935		70	9·98140	
29	0·02033		71	9·98049	
28	0·02133	100	72	9·97958	
27	0·02232		73	9·97867	
26	0·02331		74	9·97777	90
25	0·02432		75	9·97686	
24	0·02531		76	9·97596	
23	0·02630		77	9·97506	
22	0·02730		78	9·97416	
21	0·02832	102	79	9·97326	
20	0·02933		80	9·97237	

Table III.

Barometer.		
	Log.	Diff.
In. 31	0·01424	
30·9	0·01248	
8	0·01143	
7	0·01002	142
6	0·00860	
5	0·00718	
4	0·00575	
3	0·00432	144
2	0·00289	
1	0·00145	
30·0	0·00000	
29·9	9·99855	
8	9·99709	
7	9·99563	
6	9·99417	146
5	9·99270	
4	9·99123	
3	9·98075	148
2	9·98826	
1	9·98677	
29·0	9·98528	150
28·9	9·98378	
8	9·98227	
7	9·98076	
6	9·97924	
5	9·97772	152
4	9·97620	
3	9·97466	
2	9·97313	
1	9·97158	
28·0	9·97004	154